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Message from the President of American Mathematical Society

Mathematicians of the country are looking forward with pleasure to the meeting of the American Mathematical Society at Baton Rouge, beginning December 30, 1940. Those who met with the Society at the meeting of the American Association for the Advancement of Science, held at New Orleans in 1931, will want to repeat their visit to Louisiana, and will hope to greet the scholars of the South.

Mathematics represents one of the great imaginative activities of mankind. It is the language of ideas. In ordinary language, words are symbols which stand for abstractions of varying degree. In mathematics symbols are chosen which represent ideas abstractly also, but more simply because they carry in themselves and their combinations the logical connection of ideas. In difficult processes of thought no other tool is adequate.

Thus mathematics is also a most powerful tool. We hear often nowadays the statement that open frontiers are no longer geographical, but that adventure may still be sought along the march of science. In mathematics we possess the key which passes us through the gate onto these frontiers of modern life.

G. C. Evans,
University of California.

Polynomials Over Fields*

By Rufus Oldenburger
Illinois Institute of Technology

1. *Introduction.* An exposition will be given here of a new point of view in the theory of polynomials of arbitrary degree in an arbitrary number of variables. The present development sheds new light on classical problems, such as the factorability of polynomials. The theory is based on the notion of expanding a polynomial P into a sum of terms of a certain form to yield information about P that could not be readily obtained otherwise. This is roughly analogous to expanding a function $f(x)$ into a power series to obtain valuable properties of $f(x)$. The expansion of P leads to the association with P of a positive integer, called a "minimal number," which is as fundamental in the present approach as degree and number of variables.

2. *Fields.* A set K of things $a,b,c,\cdots$, which we shall call "elements" rather than "numbers," *are said to form a field* with respect to operations of "addition" $(+)$ and "multiplication" $(\cdot)$ if the following postulates are satisfied.

1. To each pair of elements a,b in the set K "corresponds" (exactly) one element $a+b$ in K.

2. To each pair of elements a,b in the set K corresponds one element $a\cdot b$ in K.

3. $a+b=b+a$.

4. $a\cdot b=b\cdot a$.

5. There is an element 0 in K such that $a+0=a$ for each a in K.

6. There is an element 1 in K such that $a\cdot 1=a$ for each a in K.

7. For each a in K there is a corresponding element $(-a)$ in K so that $a+(-a)=0$.

8. For each element $a\neq 0$ there is an element (a^{-1}) in K such that $a\cdot(a^{-1})=1$.

9. $(a+b)+c=a+(b+c)$.

10. $(a\cdot b)\cdot c=a\cdot(b\cdot c)$.

*This is the fourth article in a series of expository articles solicited by the editors.

11. $a(b+c)=ab+ac$.

12. There is at least one pair of elements a,b in K such that $a \neq b$.

We shall sometimes write $a \cdot b$ as ab, dropping the dot between a and b. We shall write $a(b^{-1})$ as a/b, whence $b^{-1}=1/b$. We shall use the notation $x^2=x \cdot x$, $x^3=(x \cdot x) \cdot x=x \cdot (x \cdot x)$, etc. Finally $a-b$ shall designate $a+(-b)$.

The real numbers are defined in such a way as to form a field. Similarly the complex numbers $a+bi$, where a,b are real numbers and $i=\sqrt{-1}$, are defined in the literature so as to form a field. These are examples of *infinite fields*, that is, we cannot count the distinct elements in these fields.

We shall give a few examples of *finite fields*, designating by the term "finite" that these fields each contain a finite number of elements. The simplest example of such a field is the field containing only the elements 0 and 1. For this field we have the following tables for addition and multiplication. It is evident that such tables can be constructed for each finite field and completely designate the field.

(1)

Addition

	0	1
0	0	1
1	1	0

Multiplication

	0	1
0	0	0
1	0	1

The addition table indicates that $0+0=0$, $0+1=1+0=1$, and $1+1=0$; the multiplication table that $0 \cdot 0=0$, $0 \cdot 1=1 \cdot 0=0$, and $1 \cdot 1=1$. The notation $1+1=0$ implies that whenever the combination $1+1$ occurs we may replace it by 0. The symbols 0 and 1 here are not to be confused with 0 and 1 of the real number system. These symbols have meaning only in that they satisfy the relations listed in the above tables. The field (1) is identical with the set of ordinary integers for which addition, subtraction, multiplication, and division are defined in the usual manner, and one writes $a=b$ if $a \equiv b \pmod{2}$. That is, 0 represents the property of "eveness" of even integers, while 1 represents the property of "oddness" of odd integers. Thus $1+1=0$ is equivalent to the statement "a sum of two odd integers is an even integer;" $1 \cdot 1=1$ is equivalent to "a product of two odd integers is odd." The reader can readily verify that Postulates $1-12$ above are satisfied for the field (1). Thus, for example, since $0+0=0$, and $1+0=1$, Postulate 5 is satisfied.

For the field defined by (1) we can simplify the following fraction in the manner indicated.

$$\frac{1+\dfrac{1+1+0}{1+1+1}}{1+1+1+1+1} = \frac{1+\dfrac{(1+1)+0}{(1+1)+1}}{(1+1)+(1+1)+1} = \frac{1+\dfrac{0+0}{0+1}}{0+0+1}$$

$$= \frac{1+\dfrac{0}{1}}{(0+0)+1} = \frac{1+0\cdot 1}{0+1} = \frac{1+0}{1} = \frac{1}{1} = 1\cdot 1 = 1.$$

For the field (1) the equation $x^2+x+1=0$ has no solution as substitution of 0 and 1 for x immediately indicates. The equation $x^2+1=0$ has the solution $x=1$ since $1^2=1$ and $1+1=0$.

We emphasize that if we have an addition table

(2)

		b	
a		c	

with entries as indicated, $c=a+b$. If (2) is a multiplication table, $c=ab$.

There is a field of 3 elements defined by the following tables.

(3)

Addition

	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Multiplication

	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

The field (3) is identical with the set of ordinary integers for which addition, subtraction, multiplication, and division are defined in the usual manner, and we write $a=b$ when $a\equiv b$ (mod. 3).

As our final example we consider the field defined by the following tables:

(4)

Addition

	0	1	ω	ϵ
0	0	1	ω	ϵ
1	1	0	ϵ	ω
ω	ω	ϵ	0	1
ϵ	ϵ	ω	1	0

Multiplication

	0	1	ω	ϵ
0	0	0	0	0
1	0	1	ω	ϵ
ω	0	ω	ϵ	1
ϵ	0	ϵ	1	ω

.

It may be noticed that ω satisfied the equation

$$\omega^2+\omega+1=0.$$

We have thus defined a field which contains the field (1) as a subset, and in which the algebraic equation

$$x^2+x+1=0$$

has a solution for x (in fact it has two) whereas this equation had no solution among the elements of the field (1).

It is to be remarked that infinite fields obviously cannot be defined by addition and multiplication tables such as those above, and must be defined in other ways.

The elements 0 and 1 are distinct for each field K. Suppose they were not. By postulate 12 above there is an element a in K such that $a\neq 0$. By Postulate 11 we have

$$a(a+0)=a\cdot a+a\cdot 0=a\cdot a+a\cdot 1,$$

whence

$$a\cdot a=a\cdot a+a.$$

Adding $-(a\cdot a)$ to each side of this equation we have

$$-(a\cdot a)+a\cdot a=-(a\cdot a)+a\cdot a+a,$$

whence $$0=0+a=a,$$

a contradiction.

If 1 occurs in the sum $(1+1+\cdots+1)$ exactly n times, we denote this sum by $n\cdot 1$. The product $(n\cdot 1)\cdot a$ we sometimes denote (for brevity) by na (here n stands for $n\cdot 1$). It follows immediately that

$$(n\cdot 1)\cdot a=(a+a+\cdots+a),$$

where a occurs in the latter sum n times. If there is a positive integer p such that $(p\cdot 1)\cdot a=0$ for each a in the given field K, and p is the smallest such integer we term p the *characteristic* of k. Otherwise, the characteristic of K is said to be *zero* (this is the zero of the natural numbers and is not to be confused with the zero element of K). Thus the characteristic of the fields of rational, real, or complex numbers is zero.

For each element a in a field K we have $a\cdot 0=0$, as is evident from the following equalities:

$$a\cdot 0=a\cdot 0+a\cdot 0-(a\cdot 0)=a\cdot(0+0)-(a\cdot 0)=(a\cdot 0)-(a\cdot 0)=0.$$

Were $ab=0$ for $a\neq 0$, $b\neq 0$, we would have $b^{-1}a^{-1}ab=0$; whence $1=0$. *Hence* $ab=0$ *implies that at least one of the elements* a,b *is zero.* Evi-

dently, if the characteristic p of K is not the zero of the natural numbers, then $p\cdot 1=0$, where 0 denotes here the zero element of K. If for an integer n, we have $n\cdot 1=0$, then $na=0a=0$ for each a in K; whence the characteristic of K is n or less. *If K has characteristic zero, K is infinite.* Then $0\cdot 1$, $1\cdot 1$, $2\cdot 1, \cdots$, are all distinct, since if $m\cdot 1=n\cdot 1$, then $(m-n)\cdot 1=0$, whence $m=n$.

Theorem 1. *If the characteristic of K is not zero it is a prime.*

Suppose that the characteristic p of K is $r\cdot s$, where r and s are positive integers not 1. Since $r<p$ and $s<p$, we have $r\cdot 1\neq 0$, $s\cdot 1\neq 0$. Let $a\neq 0$. From the definition of p, we have $(r\cdot 1)\cdot(s\cdot 1)\cdot a=0$, whence $(r\cdot 1)\cdot(sa)=0$, which is impossible since both $(r\cdot 1)$ and (sa) are $\neq 0$.

If K is a finite field (with n distinct elements), K is said to be of *order n*. Otherwise K is said to be of infinite order.

In what follows the usual symbols denoting constants and variables will be understood to designate elements belonging to a given field.

3. *Forms.* From a polynomial $P(x,y,\cdots,z)$ in the variables $x,y,\cdots,z$ we can always obtain a homogeneous polynomial

$$\mu^r P\left(\frac{x}{\mu}, \frac{y}{\mu}, \cdots, \frac{z}{\mu}\right), \tag{5}$$

where r is the degree of $P(x,y,\cdots,z)$ in all of the variables taken together. A homogeneous polynomial is called a *form*. If it contains two variables it is called *binary*, if n variables, *$n-ary$*. The study of polynomials is simplified by considering the forms (5) that can be obtained from them. For example, $P(x,y,\cdots,z)$ splits into factors $S, T,\cdots, W$ of degrees $s,t,\cdots,w$ if and only if the form (5) splits into such factors. It is to be observed that the form (5) is uniquely determined by P and that P can be obtained from (5) by setting $\mu=1$.

Thus $x^5+x^2y+z^4+1$ yields the form $x^5+x^2y\mu^2+z^4\mu+\mu^5$. The polynomial x^3+x splits into the factors $x(x^2+1)$ while its form $x^3+x\mu^2$ factors into $x(x^2+\mu^2)$.

A *power product* of $x,y,\cdots,z$ is defined to be a term of the form $x^ry^s\cdots z^t$ where $r,s,\cdots,t$ are non-negative integers. The expressions $(x+y+\cdots+z)^n$ can be expanded, as in the binomial case, into a sum $a_1f_1+a_2f_2+\cdots+a_Nf_N$ where $f_1, f_2,\cdots,f_N$ are distinct power products, and the a's are integers. A form is said to be *symmetric* with respect to a field K if it can be written as

$$k_1a_1f_1+k_2a_2f_2+\cdots+k_Na_Nf_N,$$

where the k's are in K. Thus, for a field with characteristic 2 the form x^2+xy+y^2 is not symmetric whereas $x^2+2xy+y^2=x^2+y^2$ is sym-

metric. Again, a binary quartic form is symmetric for a field K if and only if it can be written as $ax^4+4bx^3y+6cx^2y^2+4dxy^3+ey^4$, where a,b,c,d,e are elements in K.

If Q is a symmetric quadratic form in $x_1,\cdots,x_n$ we can write it as

$$\sum_{i,j=1}^{n} a_{ij}x_ix_j,$$

where $a_{ij}=a_{ji}$. Thus,

$$a_{11}x_1^2+2a_{12}x_1x_2+a_{22}x_2^2=a_{11}x_1^2+a_{12}x_1x_2+a_{21}x_2x_1+a_{22}x_2^2 \text{ where } a_{12}=a_{21}.$$

Similarly, a symmetric cubic form in $x_1,\cdots,x_n$ can be written as

$$\sum_{i,j,k=1}^{n} a_{ijk}x_ix_jx_k,$$

where $$a_{ijk}=a_{jik}=a_{jki}=a_{kij}=a_{kji}=a_{ikj}.$$

In particular,

$$\begin{aligned} a_{111}x_1^3+3a_{112}x_1^2x_2+3a_{122}x_1x_2^2+a_{222}x_2^3 \\ =a_{111}x_1^3+a_{112}x_1^2x_2+a_{121}x_1^2x_2+a_{122}x_1x_2^2+a_{211}x_1^2x_2 \\ +a_{212}x_1x_2^2+a_{221}x_1x_2^2+a_{222}x_2^3, \end{aligned}$$

where $a_{112}=a_{121}=a_{211}$, and $a_{122}=a_{212}=a_{221}$. In general, we write a symmetric $p-ic$ form F in $x_1,\cdots,x_n$ as

$$\sum_{i,j,\ldots,m=1}^{n} a_{ij\cdots m}x_ix_j\cdots x_m, \tag{6}$$

where the value of $a_{ij\cdots m}$ is unchanged by permuting the subscripts of this element. The set of coefficients in (6) then form what is termed a *symmetric matrix* $(a_{ij\cdots m})$. If F is quadratic the associated matrix (a_{ij}) can be written as

$$\left\| \begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{matrix} \right\| . \tag{7}$$

The index i is called the *row index* in (7), and j the *column index*. Since the matrix $(a_{ij\cdots m})$ of F in (6) has p indices $i,j,\cdots,m$ it is said to be $p-$way and can be displayed in a manner analogous to (7) in $p-$dimensional space. Since the indices of $(a_{ij\cdots m})$ range over $1,2,\cdots,n$ we say that this matrix is of *order n*. For example,

$$x_1^2+6x_1x_2+4x_1x_3+x_3^2$$

has the symmetric matrix

$$\begin{Vmatrix} 1 & 3 & 2 \\ 3 & 0 & 0 \\ 2 & 0 & 1 \end{Vmatrix}$$

of order 3.

The matrix (7) is a special case of the general 2-way matrix:

$$\begin{Vmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{Vmatrix} \tag{8}$$

with n rows and m columns. Let M be an array of elements obtained from (8) by striking out certain rows and certain columns from (8). It is called a *minor* of (8). If M is square (same number of rows as columns) we can take the ordinary determinant $|M|$ of M. It is called a *determinant minor* of (8). If $|M|$ has r rows and r columns it is said to be of *order* r. In any given matrix (8) there is a non-vanishing determinant with largest order r. This order is termed the *rank* of (8). If the *rank* of a square matrix equals its *order* the matrix is said to be *non-singular*. The determinant of the matrix is then non-zero. Thus

$$\begin{Vmatrix} 1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 3 \\ 2 & 2 & 2 & 6 \end{Vmatrix}$$

is of rank 2 whereas

$$\begin{Vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{Vmatrix}$$

is of rank 3, order 3, and is non-singular.

4. *Essential variables.* A set of substitutions

$$\begin{aligned} x_1 &= b_{11}y_1 + b_{12}y_2 + \cdots + b_{1n}y_n, \\ x_2 &= b_{21}y_1 + b_{22}y_2 + \cdots + b_{2n}y_n, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_n &= b_{n1}y_1 + b_{n2}y_2 + \cdots + b_{nn}y_n, \end{aligned} \tag{9}$$

with coefficients in a field K and a non-singular matrix

$$B = \begin{Vmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{Vmatrix}$$

is called a ***non-singular linear transformation.*** Since B is non-singular we can solve (9) for the y's in terms of the x's. Applying the transformation (9) to the x's in the form F given by (6), we obtain a symmetric form F' in $y_1, \cdots, y_n$ which can be written as

$$c_{qr\cdots s}\, y_q y_r \cdots y_s,$$

where

$$c_{qr\cdots s} = \sum_{i,j,\ldots,m=1}^{n} a_{ij\ldots m} b_{iq} b_{jr} \cdots b_{ms}.$$

Transformations of type (9) are important because we can often choose (9) so that F goes into a form F' which can be more easily studied than F while the algebraic properties of F in which we shall be interested are preserved under such transformations. We note that *if $F = Q^q R^r \cdots T^t$, where $Q, R, \cdots, T$ are forms of degrees $q, r, \cdots, t$ respectively, $F' = Q_1{}^q R_1{}^r \cdots T_1{}^t$, where $Q_1, R_1, \cdots, T_1$ are of degrees $q, r, \cdots, t$, respectively.* All of the coefficients in $Q, R, \cdots, T$ and (9) are assumed here to be in some given field K, whence it follows that the coefficients in $Q_1, R_1, \cdots, T_1$ are also in K. In particular, F' is of degree p since T is of degree p. The integer p is said to be *invariant* under transformations of type (9).

We illustrate the above remarks with a few obvious examples. The form $Q = (x+y+z)^2 + (x-y-z)^2$ can be transformed by the substitutions

$$\begin{aligned} u &= x+y+z, \\ v &= x-y-z, \\ z' &= z, \end{aligned} \tag{10}$$

into $Q' = u^2 + v^2$. Since Q' does not split into linear factors in the field of rational numbers and the coefficients in (10) are rational, it follows that the original form Q does not split up into linear factors for this field. However, the form $E = (x+y+z)^2 - (x-y-z)^2$ can be transformed by (10) into $E' = u^2 - v^2$. Obviously E and E' split into linear factors with coefficients in the field of rational numbers.

We note that the number of variables in a form F is not necessarily invariant under transformations (9). Thus Q above in 3 variables goes into a form Q' with 2 variables under the transformation (10). It is easy to see that Q' cannot be transformed by a transformation (9) into a form with less than 2 variables. More generally, among all of the forms into which a given form F can be transformed by (9) there is a form F' with a minimum number t of variables, that is, F cannot be transformed by a non-singular linear transformation (9) to

a form with less variables. F' is not unique. From the manner in which we have defined t, it is clear that t is *invariant* under non-singular linear transformations. We shall say that t of the variables in F are *essential*. If $t=n$, n being the number of variables in F, all of the variables in F are essential. The number t may be determined for a particular form F in a manner which will now be described.

We write the matrix $(a_{ij\cdots m})$ of the form F defined by (6) as a 2-way matrix $(a_{i,j\cdots m})$ with i as row index, and $j,\cdots,m$ as a composite column index. For example, the matrix $(a_{i,jm})$ of order 2 can be written as

$$\begin{Vmatrix} a_{111} & a_{112} & a_{121} & a_{122} \\ a_{211} & a_{212} & a_{221} & a_{222} \end{Vmatrix} . \tag{11}$$

The rank of $(a_{i,ij\cdots m})$ is termed the *principal determinant rank* of F, abbreviated *p. d. rank*. The term "principal" is merely a term used to distinguish this rank from other known ranks of F which are defined in terms of higher dimensional determinants. An example on p. d. ranks is given near the end of this section.

We shall need to use a few notions from ordinary matrix theory. The *transpose* A' of the matrix A given in (8) is defined to be the matrix

$$\begin{Vmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \cdot & \cdot & \cdots & \cdot \\ a_{1m} & a_{2m} & \cdots & a_{nm} \end{Vmatrix}$$

obtained from A by interchanging rows and columns of A. Thus the transpose of

$$\begin{Vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{Vmatrix}$$

is

$$\begin{Vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{Vmatrix} .$$

The *product* AB of the matrix A of (8) and

$$B=\begin{Vmatrix} b_{11} & b_{12} & \cdots & b_{1r} \\ b_{21} & b_{22} & \cdots & b_{2r} \\ \cdot & \cdot & \cdots & \cdot \\ b_{m1} & b_{m2} & \cdots & b_{mr} \end{Vmatrix}$$

is defined to be the matrix

$$AB=\left\|\begin{matrix} \left(\sum_{\alpha=1}^{m} a_{1\alpha}b_{\alpha 1}\right) & \left(\sum_{\alpha=1}^{m} a_{1\alpha}b_{\alpha 2}\right) & \cdots & \left(\sum_{\alpha=1}^{m} a_{1\alpha}b_{\alpha r}\right) \\ \left(\sum_{\alpha=1}^{m} a_{2\alpha}b_{\alpha 1}\right) & \left(\sum_{\alpha=1}^{m} a_{2\alpha}b_{\alpha 2}\right) & \cdots & \left(\sum_{\alpha=1}^{m} a_{2\alpha}b_{\alpha r}\right) \\ \cdot & \cdot & \cdots & \cdot \\ \left(\sum_{\alpha=1}^{m} a_{n\alpha}b_{\alpha 1}\right) & \left(\sum_{\alpha=1}^{m} a_{n\alpha}b_{\alpha 2}\right) & \cdots & \left(\sum_{\alpha=1}^{m} a_{n\alpha}b_{\alpha r}\right) \end{matrix}\right\|$$

with n rows and r columns. The notation

$$\sum_{\alpha=1}^{m} a_{2\alpha}b_{\alpha 1}$$

means $$a_{21}b_{11}+a_{22}b_{21}+\cdots+a_{2m}b_{m1}.$$

Thus $$\left\|\begin{matrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{matrix}\right\| \cdot \left\|\begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix}\right\| = \left\|\begin{matrix} 3 & -1 \\ 1 & -1 \\ 2 & 0 \end{matrix}\right\| .$$

If A and B are matrices and if we write AB, it is to be understood that the number of columns in A equals the number of rows in B.

The *direct product* $A\times B\times\cdots\times C$ of a set of matrices

$$A=(a_{\alpha i}),\quad B=(b_{\beta j}),\cdots,\ C=(c_{\delta m})$$

is by definition the matrix

(12) $$(d_{\alpha\beta\cdots\delta,ij\cdots m}),$$

where $$d_{\alpha\beta\cdots\delta ij\cdots m}=a_{\alpha i}b_{\beta j}\cdots c_{\delta m},$$

and (12) is obtained by multiplying the elements of $A,B,\cdots,C$ in all possible ways using one element from each matrix in each product (cf. the example below), and further we take $\alpha,\beta,\cdots,\delta$ as composite row index and $i,j,\cdots,m$ as composite column index. Thus the direct product $A\times B$ of

$$A=\left\|\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}\right\|, \qquad B=\left\|\begin{matrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{matrix}\right\|$$

is

$$A\times B=\left\|\begin{matrix} a_{11}b_{11} & a_{11}b_{12} & a_{11}b_{13} & a_{12}b_{11} & a_{12}b_{12} & a_{12}b_{13} \\ a_{11}b_{21} & a_{11}b_{22} & a_{11}b_{23} & a_{12}b_{21} & a_{12}b_{22} & a_{12}b_{23} \\ a_{21}b_{11} & a_{21}b_{12} & a_{21}b_{13} & a_{22}b_{11} & a_{22}b_{12} & a_{22}b_{23} \\ a_{21}b_{21} & a_{21}b_{22} & a_{21}b_{23} & a_{22}b_{21} & a_{22}b_{22} & a_{22}b_{23} \\ a_{31}b_{11} & a_{31}b_{12} & a_{31}b_{13} & a_{32}b_{11} & a_{32}b_{12} & a_{32}b_{13} \\ a_{31}b_{21} & a_{31}b_{22} & a_{31}b_{22} & a_{32}b_{21} & a_{32}b_{22} & a_{32}b_{23} \end{matrix}\right\| .$$

Again, if $A = \left\| \begin{matrix} 1 & -1 \\ 1 & 1 \end{matrix} \right\|, \quad B = \left\| \begin{matrix} 1 & 2 \\ 2 & 1 \end{matrix} \right\|,$

then
$$A \times B = \left\| \begin{matrix} 1 & 2 & -1 & -2 \\ 2 & 1 & -2 & -1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \end{matrix} \right\|,$$

and
$$B \times B = \left\| \begin{matrix} 1 & 2 & 2 & 4 \\ 2 & 1 & 4 & 2 \\ 2 & 4 & 1 & 2 \\ 4 & 2 & 2 & 1 \end{matrix} \right\|.$$

Computation reveals that A, B, $A \times B$ and $B \times B$ are all non-singular. This illustrates the Theorem 2 to follow.

If we solve (9) for the y's in terms of the x's, we obtain a set of equations

$$\begin{aligned} y_1 &= B_{11}x_1 + B_{12}x_2 + \cdots + B_{1n}x_n, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ y_n &= B_{n1}x_1 + B_{n2}x_2 + \cdots + B_{nn}x_n. \end{aligned} \tag{13}$$

The matrix
$$\left\| \begin{matrix} B_{11} & B_{12} & \cdots & B_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ B_{n1} & B_{n2} & \cdots & B_{nn} \end{matrix} \right\|$$

of coefficients of (13) is called the *inverse* of B, denoted by B^{-1}. We note that $BB^{-1} = I$, where

$$I = \left\| \begin{matrix} 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{matrix} \right\|.$$

The matrix I is called the *identity* matrix of order n.

Thus the inverse of
$$\left\| \begin{matrix} 0 & 1 \\ 1 & 1 \end{matrix} \right\|$$

is
$$\left\| \begin{matrix} -1 & 1 \\ 1 & 0 \end{matrix} \right\|,$$

and we observe that

$$\left\| \begin{matrix} 0 & 1 \\ 1 & 1 \end{matrix} \right\| \cdot \left\| \begin{matrix} -1 & 1 \\ 1 & 0 \end{matrix} \right\| = \left\| \begin{matrix} -1 & 1 \\ 1 & 0 \end{matrix} \right\| \cdot \left\| \begin{matrix} 0 & 1 \\ 1 & 1 \end{matrix} \right\| = \left\| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right\|.$$

Theorem 2. *The direct product of non-singular matrices is non-singular.*

Let A,B be non-singular matrices of orders n and m respectively, and let I_n, I_m be identity matrices of orders n and m respectively. Then we write $A=(a_{\alpha i})$, $B=(b_{\beta j})$, $A^{-1}=(A_{ir})$, $B^{-1}=(B_{js})$, $I_n=(\delta_{\alpha r})$, $I_m=(\delta_{\beta s})$, whence we have

$$\left(\sum_{i=1}^{n}\ \sum_{j=1}^{m}\ a_{\alpha i}b_{\beta j}A_{ir}B_{js}\right)$$

$$=\left(\left[\sum_{i=1}^{n}\ a_{\alpha i}A_{ir}\right]\left[\sum_{j=1}^{m}\ b_{\beta i}B_{js}\right]\right)=(\delta_{\alpha r}\delta_{\alpha s}).$$

Consequently we have

$$(A\times B)\cdot(A^{-1}\times B^{-1})=(AA^{-1})\times(BB^{-1})=I_n\times I_m. \tag{14}$$

But $I_n\times I_m=I_{nm}$, where I_{nm} is the identity matrix of order nm. From determinant theory we have the relation $|AB|=|A|\cdot|B|$ for square matrices A,B. Since I_{nm} is non-singular (its determinant is 1), and $|I_{nm}|=|A^{-1}\times B^{-1}|\cdot|A\times B|$, the matrix $A\times B$ is non-singular. The Theorem follows by induction.

Theorem 3. *The number of essential variables in a symmetric form F is equal to the p. d. rank of F.*

As earlier in this section we associate with F, given by (6), the 2-way matrix $A_2=(a_{i,j\cdots m})$ and with a form $F'=c_{qr\cdots s}y_qy_r\cdots y_s$, obtained from F by the transformation (9), the 2-way matrix

$$C_2=(c_{q,r\cdots s}).\quad \text{Then}$$

$$B'A_2(B\times\cdots\times B)=C_2, \tag{15}$$

where the matrix $(B\times\cdots\times B)$ in (15) is the direct product of $(p-1)B's$, and where B is the matrix of coefficients of (9). We illustrate (15) with an example. We let F be the binary cubic

$$\sum_{i,j,k=1}^{2} a_{ijk}x_ix_jx_k.$$

Applying the transformation (9) with $n=2$ to F, we obtain

$$F'=\sum_{q,r,s=1}^{2} C_{qrs}y_qy_ry_s,$$

where

$$C_{qrs}=\sum_{i,j=1}^{2}\ a_{ijk}b_{iq}b_{jr}b_{ks}.$$

Now A_2 is given by (11), and

$$C_2=\left\| \begin{matrix} c_{111} & c_{112} & c_{121} & c_{122} \\ c_{211} & c_{212} & c_{221} & c_{222} \end{matrix} \right\|, \quad B'=\left\| \begin{matrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{matrix} \right\|$$

$$B\times B=\left\| \begin{matrix} b_{11}b_{11} & b_{11}b_{12} & b_{12}b_{11} & b_{12}b_{12} \\ b_{11}b_{21} & b_{11}b_{22} & b_{12}b_{21} & b_{12}b_{22} \\ b_{21}b_{11} & b_{21}b_{12} & b_{22}b_{11} & b_{22}b_{12} \\ b_{21}b_{21} & b_{21}b_{22} & b_{22}b_{21} & b_{21}b_{22} \end{matrix} \right\| .$$

Evidently these matrices satisfy (15).

We need here two lemmas from ordinary matrix theory.

Lemma 1. *The rank of AB and of BA where B is non-singular equals the rank of A.*

Lemma 2. *If A is of rank r we can find a non-singular matrix B such that for each non-singular D,*

$$BAD=\left\| \begin{matrix} A' \\ 0 \end{matrix} \right\| ,$$

where A' is a minor with r rows.

Lemma 1 is proved in the literature by showing that each determinant minor of AB or BA of order p is a sum of products of determinant minors of A and B of order p. Lemma 2 is proved by showing that there is a non-singular matrix B such that all but the first r rows of BA consist of zero elements. Simple computation reveals that multiplication of AB by D on the right brings the rows of zeros in AB into rows of zeros in ABD.

Let the rank of A_2 be denoted by g, and the number of essential variables in F by t. By Lemma 2 we can find a non-singular matrix B' such that C_2 of (15) has $n-g$ rows of zeros. Hence F' involves only g variables from the set $y_1,\cdots,y_n$. Thus $g\geqq t$. Now let F be transformed to a form F' which involves only t variables. In C_2 there are $(n-t)$ rows of zeros, whence the rank of C_2 is not greater than t. By Lemma 1 the ranks of A_2 and C_2 are equal; whence $t\geqq g$. Hence $t=g$, and the theorem is proved.

As an example, we consider the form $c=x^3+3xy^2+3xz^2+6xyz$, where we have written x,y,z for x_1,x_2,x_3 respectively. We assume that the field of coefficients has characteristic $\neq 2$. The 2-way matrix $(a_{i,jk})$ of C is

$$\begin{array}{c|ccccccccc|} & xx & xy & xz & yx & yy & yz & zx & zy & zz \\ x & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ y & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ z & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \end{array} ,$$

which is of rank 2, whence the p. d. rank of C is 2. Actually

$$C = \tfrac{1}{2}[(x+y+z)^3+(x-y-z)^3]\ ,$$

whence C can be transformed by

$$u = x+y+z,$$

$$v = x-y-z,$$

$$z' = z,$$

into $\tfrac{1}{2}u^2+\tfrac{1}{2}v^2$ in 2 essential variables u, v.

To carry through the argument of Theorem 3 we have made the implicit assumption that the field K in which we are working is such that if two symmetric forms

$$\sum a_{ij\cdots m}x_ix_j\cdots x_m \text{ and } \sum b_{ij\cdots m}x_ix_j\cdots x_m$$

of degree p in n variables are equal for all values of the x's in K, then corresponding coefficients are equal, that $a_{ij\cdots m}=b_{ij\cdots m}$. We have recently proved that *the field K has this property if and only if it is of order $p+1$ or more and its characteristic does not divide the coefficients in the expansion of* $(x_1+x_2+\cdots+x_m)^p$. For the sake of brevity the proof of this will be omitted. If the characteristic of K is zero K has this property.

5. *Representations.* The theory of forms would be much simpler if each form of degree p could be written as aL^p where L is a linear form and a is an element in the given field. That this is in general impossible is obvious. One is naturally led to ask: Can F be written as a sum of such terms? We answer this question in the affirmative in Theorem 4. Suppose that

$$F = aL^p+bM^p+\cdots+dN^p,$$

where $L,M,\cdots,N$ are linear forms with coefficients in a field K, and $a,b,\cdots,d$ are elements in K. Then we shall say that *F is a linear combination of p-th powers of linear forms with respect to K.*

We shall need the following lemma where $C(p+q,p)$ denotes the number of combinations of $p+q$ things taken p at a time.

Lemma 3. *Let K be a field of order r, where $r\geqq p+q$, and let $C(p+q,p)$ be denoted by k. For K the monomial kx^py^q can be written as a linear combination of $(p+q)$-th powers of $(p+q+1)$ linear forms.*

We write

$$kx^py^q \equiv \alpha_1 y^{p+q}+\sum_{i=2}^{p+q+1}\alpha_i(x+b_iy)^{p+q}.$$

Equating corresponding coefficients we obtain the following equations which are a set of $(p+q+1)$ non-homogeneous linear equations in the $(p+q+1)\alpha$'s.

$$(16) \qquad \sum_{i=2}^{p+q+1} b_i{}^r\alpha_i=0, \quad r=0,1,\cdots,q-1, \quad q+1,\cdots,q+p-1,$$

$$\sum_{i=2}^{p+q+1} b_i^q\alpha_i=1, \quad \alpha_1+\sum_{i=2}^{p+q+1} b_i^{q+p}\alpha_i=0.$$

The determinant of coefficients of (16) is

$$\begin{vmatrix} 0 & 1 & 1 & \cdot & \cdot & \cdot & 1 \\ 0 & b_2 & b_3 & \cdot & \cdot & \cdot & b_{p+q+1} \\ 0 & b_2{}^2 & b_3{}^2 & \cdot & \cdot & \cdot & b^2_{p+q+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & b_2^{p+q-1} & b_3^{p+q-1} & \cdot & \cdot & \cdot & b_{p+q+1}^{p+q-1} \\ 1 & b_2^{p+q} & b^{p+q} & \cdot & \cdot & \cdot & b_{p+q+1}^{p+q} \end{vmatrix},$$

which evidently equals

$$(17) \qquad \begin{vmatrix} 1 & 1 & \cdot & \cdot & \cdot & 1 \\ b_2 & b_3 & \cdot & \cdot & \cdot & b_{p+q+1} \\ b_2{}^2 & b_3{}^2 & \cdot & \cdot & \cdot & b^2_{p+q+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_2^{p+q-1} & b_3^{p+q-1} & \cdot & \cdot & \cdot & b_{p+q+1}^{p+q-1} \end{vmatrix}$$

The determinant (17) is the well-known *Vandermonde determinant* which occurs so often in algebraic literature. It is easily verified that this determinant is a product of all of the differences (b_i-b_j) where $i>j$, and $i,j=1,2,\cdots,p+q+1$. Since K is of order at least $p+q$ we can choose the b's distinct, whence (17) is not zero, and the system (16) is solvable for the α's. We are now ready for our basic theorem.

Theorem 4. *For a field K of order p or more, a symmetric form of degree p is a linear combination of p-th powers of linear forms.*

The theorem is trivially true for a form F in one variable, for F is simply ax^p. By Lemma 3 the theorem is true for a symmetric form of degree p in two variables because each such form is a sum of terms of the type $aC(p,q)x^{p-q}y^q$. We proceed with an induction process. We assume that the theorem is true for forms in fewer than n variables. A symmetric form F of degree p in n variables $x,y,\cdots,z,w$ is a linear combination of terms of the form $T=kx^ry^s\cdots z^tw^q$, where

$$r+s+\cdots+t+q=p, \qquad k=\frac{p!}{r!s!\cdots t!q!}\ .$$

It is evidently sufficient to consider monomials T only.. By assumption we have for K

$$fx^ry^s\cdots z^t=\sum_{i=1}^{\rho}\mu_iM_i^{p-q},$$

where $M_1, M_2,\cdots, M_\rho$ are linear forms and

$$f=\frac{(p-q)!}{r!s!\cdots t!}\ .$$

The constants k and f are evidently the coefficients of $x^ry^s\cdots z^tw^q$ and $x^ry^s\cdots z^t$ in the expansions of $(x+y+\cdots+z+w)^p$ and $(x+y+\cdots+z)^{p-q}$ respectively. Now

$$k=f\cdot C(p,q).$$

Hence
$$T=\sum_{i=1}^{\rho}\mu_iC(p,q)M_i^{p-q}w^q.$$

By Lemma 3 we have for K

$$C(p,q)M_i^{p-q}w^q=\sum_{j=1}^{\sigma}\psi_{ij}L_{ij}^p,$$

where the L's are linear forms; for we can think of M_i as a single variable. Thus

$$T=\sum_{i=1}^{\rho}\sum_{j=1}^{\rho}\lambda_{ij}L_{ij}^p, \tag{18}$$

where
$$\lambda_{ij}=\mu_i\psi_{ij}.$$

Since each term in F is of degree p and can be written for K as a sum (18), F itself can be written as such a sum.

It is to be emphasized that the result in Theorem 4 *depends on the order of the field and not on its characteristic.* Also, *the result does not depend on the number of variables in F.*

We proceed with a few illustrations. Let K be a field with characteristic not 2. Then $6x^2y^2$ has the representation

$$1/2(x+y)^4+1/2(x-y)^4-x^4-y^4$$

for K.

For the field K composed of the elements 0,1 and characteristic 2 we cannot write $3x^2y\equiv x^2y$ as a linear combination of cubes of linear forms. Thus Theorem 4 cannot be strengthened to include fields of

order less than p. However, for the field K of rational numbers (in this case order $= \infty$) we can write

$$3x^2y = {}^1/_2(x+y)^3 - {}^1/_2(x-y)^3 - y^3.$$

For a field with characteristic not 2 we can write

$$2xy = {}^1/_2(x+y)^2 - {}^1/_2(x-y)^2.$$

6. Minimal numbers. Among all of the representations of a form F for a field K there is a representation involving a least number of terms. We shall denote this number by $m(F)$, termed the *minimal number* of F with respect to K. The associated representation is called a *minimal representation*. The proof of the theorem which follows will appear elsewhere in the literature and is omitted here.

Theorem 5. *Let K be a field such that if two forms of degree p in n variables are identically equal, corresponding coefficients are equal. Let F be a form of degree p in n essential variables, and let N be the number of terms in the formal expansion of $(x_1+x_2+\cdots+x_n)^p$. The minimal number of F with respect to K satisfies the inequalities*

$$n \leqq m(F) \leqq N.$$

Corollary 1. *For K as defined in Theorem* 5 *with $n=2$, the minimal number of a binary form of degree p satisfies*

$$m(F) \leqq p+1.$$

Corollary 2. *Let K be a field with characteristic different from* 2,3. *Let F be a binary cubic form. Then*

$$m(F) \leqq 4.$$

The following theorem is valid for fields as defined in Theorem 5.

Theorem 6. *The minimal number of a form F is invariant under non-singular linear transformations.*

If
$$F = aL^p + bM^p + \cdots + cN^p,$$
a transformation of type (9) brings F into a form
$$F' = aL_1{}^p + bM_1{}^p + \cdots + cN_1{}^p,$$
where $L_1, \cdots, N_1$ are linear. Theorem 6 is thus proved.

The p. d. rank of a symmetric quadratic form is the same as the rank of this form as defined in the literature. In the following theorem the field is assumed to have characteristic not 2.

Theorem 7. *The minimal number of a quadratic form equals the rank of the form.*

We shall need from the literature the following lemma, which we present without proof.

Lemma 4. *A symmetric quadratic form Q of rank r can be brought by a non-singular linear transformation into*

$$Q' = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_r y_r^2.$$

Since the y's are linear forms in the variables of Q we can substitute these forms for the y's in Q' to return to our form Q, obtaining

$$Q = \lambda_1 L_1^2 + \lambda_2 L_2^2 + \cdots + \lambda_r L_r^2.$$

Hence $m(Q) \leqq r$. If now

$$Q = \mu_1 M_1^2 + \mu_2 M_2^2 + \cdots + \mu_t M_t^2,$$

where $t < r$ we can write

$$y_1 = M_1, \quad y_2 = M_2, \cdots, y_t = M_t$$

so that

$$Q = \mu_1 y_1^2 + \mu_2 y_2^2 + \cdots + \mu_t y_t^2.$$

Then the number of essential variables in Q is t or less. Consequently by Theorem 3, $t \geqq r$. Hence $t = r$, and Theorem 7 is proved.

Thus *for quadratic forms, rank = p. d. rank = number of essential variables = minimal number.*

For quadratic forms the minimal number does not depend on K, but this is not true of forms of higher degree. In particular we can prove that $m(6x^2y^2) = 3$ if the field K contains an element $\omega \neq 1$ such that $\omega^3 = 1$, and $m(6x^2y^2) = 4$ if there is no such element ω. We assume here that $2 \neq 0$, $3 \neq 0$ for K.

7. Minimal numbers of sums and products of forms. The sum $F+G$ of symmetric forms F, G which are of the same degree is evidently symmetric, whence $F+G$ has a minimal number if F and G have minimal numbers.

Theorem 8. *Let K be a field of order at least p, and let F and G be symmetric forms of degree p. For the minimal numbers of F, G and $F+G$ with respect to K we have*

$$m(F) + m(G) \geqq m(F+G) \geqq |m(F) - m(G)|. \tag{19}$$

If $m(F) = \rho$, $m(G) = \sigma$, we have

$$F = \sum_{i=1}^{\rho} \lambda_i L_i^p, \quad G = \sum_{i=1}^{\sigma} \mu_i M_i^p,$$

$$F + G = \sum_{i=1}^{\rho} \lambda_i L_i^p + \sum_{i=1}^{\sigma} \mu_i M_i^p,$$

whence the left inequality in (19) follows. Without loss of generality, we may assume that $m(F)\geqq m(G)$.

Now $\qquad m(-G)=m(G)$, since $-G=\sum_{i=1}^{\sigma}(-\mu_i)M_i^p$.

By the left inequality in (19)

$$m(F+G)+m(-G)\geqq m(F).$$

Hence $\qquad m(F+G)\geqq m(F)-m(G)=|m(F)-m(G)|$.

Corollary 1. *Let r,s,t be the ranks of symmetric quadratic forms Q, Q', and $Q+Q'$, respectively. Then $r+s\geqq t\geqq |r-s|$.*

Corollary 2. *Addition of a term aL^p, where L is linear, to a form F of degree p yields a new form F_1 whose minimal number differs from that of F by at most* 1.

Corollary 2 follows from Theorem 8 since $m(G)=1$ when $G=aL^p$ and $a\neq 0$. Theorems 9-12 below are valid for fields for which equality of forms implies equality of corresponding coefficients.

Theorem 9. *Let forms F and G be of degree p and q, respectively. There is the following relation between minimal numbers*:

$$m(FG)\leqq(p+q+1)m(F)m(G).$$

We have $\qquad F=\sum_{i=1}^{\rho}\lambda_i L_i^p,\quad G=\sum_{j=1}^{\sigma}\mu_j M_j^q,$

whence

$$FG=\sum_{i=1}^{\rho}\sum_{j=1}^{\sigma}\lambda_i\mu_j L_i^p M_j^q. \tag{20}$$

By Lemma 3, the product $L_i^pM_j^q$ can be written as a linear combination of $p+q$-th powers of $(p+q+1)$ linear forms. Since each term in (20) can be so expanded, the theorem is proved.

Theorem 10. *The minimal number of x^py^q exceeds both p and q.*

The minimal number of x^py^q is the same as the minimal number of kx^py^q, where $k=C(p+q,p)$. Suppose that $p\geqq q$. We write

$$kx^py^q=\sum_{i=1}^{\rho}\alpha_i(a_ix+y)^{p+q},\quad \rho\leqq p. \tag{21}$$

In order that (21) be minimal it is necessary that the a's be distinct. For if $a_1=a_2$, we have

$$\alpha_1(a_1x+y)^{p+q}+\alpha_2(a_1x+y)^{p+q}=(\alpha_1+\alpha_2)(a_1x+y)^{p+q}$$

and $kx^p y^q$ can be written as a sum with less than ρ terms. Equating the coefficients of $y^{p+q}, xy^{p+q-1}, \cdots, x^{\rho-1}y^{p+q-\rho+1}$ we obtain the equations

$$
(22)\qquad
\begin{array}{llll}
\alpha_1 & +\ \alpha_2 & +\cdots+ & \alpha_\rho = 0, \\
a_1\alpha_1 & +\ a_2\alpha_2 & +\cdots+ & a_\rho\alpha_\rho = 0, \\
a_1^2\alpha_1 & +\ a_2^2\alpha_2 & +\cdots+ & a_\rho^2\alpha_\rho = 0, \\
\cdot & \quad\cdot & \cdots & \cdot\ \ \cdot \\
a_1^{\rho-1}\alpha_1 & +\ a_2^{\rho-1}\alpha_2 & +\cdots+ & a_\rho^{\rho-1}\alpha_\rho = 0,
\end{array}
$$

which is a system of ρ homogeneous equations in ρ α's. From the elementary theory of linear equations (22) has a solution if and only if the determinant of coefficients vanishes. This determinant is the Vandermonde determinant

$$
\begin{vmatrix}
1 & 1 & \cdots & 1 \\
a_1 & a_2 & \cdots & a_\rho \\
a_1^2 & a_2^2 & \cdots & a_\rho^2 \\
\cdot & \cdot & \cdots & \cdot \\
a_1^{\rho-1} & a_2^{\rho-1} & \cdots & a_\rho^{\rho-1}
\end{vmatrix},
$$

which is a product of differences between the a's. Since these are distinct this determinant does not vanish, whence (22) cannot be solved.

We now write

$$
(23)\qquad kx^p y^q = \sum_{i=1}^{\rho-1} \alpha_i(a_i x + y)^{p+q} + \alpha_\rho x^{p+q}.
$$

As above, the a's in (23) must be distinct if (23) is minimal. Equating the coefficients of y^{p+q}, $xy^{p+q-1}, \cdots, x^{\rho-2}y^{p+q-\rho+2}$, we obtain the equations

$$
(24)\qquad
\begin{array}{llll}
\alpha_1 & +\ \alpha_2 & +\cdots+ & \alpha_{\rho-1} = 0, \\
a_1\alpha_1 & +\ a_2\alpha_2 & +\cdots+ & a_{\rho-1}\alpha_{\rho-1} = 0, \\
\cdot & \quad\cdot & \cdots & \cdot \\
a_1^{\rho-2}\alpha_1 & +\ a_2^{\rho-2}\alpha_2 & +\cdots+ & a_{\rho-1}^{\rho-2}\alpha_{\rho-1} = 0,
\end{array}
$$

which is a system exactly like (22) with ρ replaced by $\rho-1$. Hence there is no solution of (24) for the α's.

Theorem 10 is now proved. For if $kx^p y^q$ has a minimal representation, it has a representation of the form (21) or (23); this is proved as follows. The most general representation of $kx^p y^q$ is of the type

$$
(25)\qquad kx^p y^q = \alpha_1(a_1 x + b_1 y)^{p+q} + \alpha_2(a_2 x + b_2 y)^{p+q} + \cdots + \alpha_\rho(a_\rho x + b_\rho y)^{p+q}.
$$

If two of the a's are zero, we may suppose that these are a_1 and a_2. The sum of the first two terms in the representation (25) is then $\alpha_1(b_1y)^{p+q}+\alpha_2(b_2y)^{p+q}$, which is the same as $\alpha_1'y^{p+q}$, where

$$\alpha_1'=\alpha_1b_1^{p+q}+\alpha_2b_2^{p+q}.$$

Thus the first two terms in (25) may be combined into one and (25) is not minimal as written. One reasons similarly if two b's vanish. If no b vanishes we can write

$$\alpha_1(a_1x+b_1y)^{p+q}=\alpha_1'(a_1'x+y)^{p+q},$$

where
$$\alpha_1'=\alpha_1b_1^{p+q},\qquad a_1'=\frac{a_1}{b_1}\cdot$$

Thus (25) reduces to type (21). If one b vanishes, say $b_\rho=0$, we obtain an expression (23) provided we write

$$\alpha_\rho'=\alpha_\rho a_\rho^{p+q}.$$

Theorem 11. *If a form F of degree p is a constant times a product of powers of two linear forms, the minimal number of F is greater than $p/2$ or $(p+1)/2$ according as p is even or odd.*

Let L and M be linear forms such that $L\neq kM$ and $M\neq kL$ for each k. L and M are then said to be independent. Substituting $x=L$, $y=M$ in x^ry^s we obtain L^rM^s. By the invariance of the minimal number, $m(x^ry^s)=m(L^rM^s)$. For any decomposition of p into a sum $r+s$, either r or s is at least $p/2$ or $(p+1)/2$ according as p is even or odd. Thus if $F=aL^rM^s$, by Theorem 10, $m(F)>p/2$ or

$$m(F)>(p+1)/2$$

as the case may be.

As an example we propose the question: Except for a constant factor does the form

$$Q=x^5+y^5+(x+y)^5$$

factor in the field of rational numbers into a product of powers of not more than two linear forms?

Obviously, $m(Q)\leqq 3$. Simple computation reveals that

$$Q\neq\lambda(ax+by)^5+\mu(cx+dy)^5,$$

whence $m(Q)=3$. Since $m(aL^5)=1$, Q is not a constant times the power of a linear form. Now $m(aL^4M)^3$, $m(aL^3M^2)>3$; whence Q is not a constant times a product of powers of two independent linear forms.

Theorem 12. *The minimal number of $x^p y$ with respect to a field K with characteristic zero is $p+1$.*

We write

$$(p+1)x^p y = \sum_{i=1}^{p+1} \alpha_i (a_i x + y)^{p+1}.$$

Equating coefficients, we obtain the following system of linear equations:

$$(26) \quad \begin{array}{llll}
\alpha_1 & + \alpha_2 & + \cdots + & \alpha_{p+1} = 0, \\
a_1\alpha_1 & + a_2\alpha_2 & + \cdots + & a_{p+1}\alpha_{p+1} = 0, \\
a_1^2\alpha_1 & + a_2^2\alpha_2^2 & + \cdots + & a_{p+1}^2\alpha_{p+1} = 0, \\
\cdot & \cdot & \cdots & \cdot \\
a_1^{p-1}\alpha_1 & + a_2^{p-1}\alpha_2 & + \cdots + & a_{p+1}^{p-1}\alpha_{p+1} = 0, \\
a_1^{p}\alpha_1 & + a_2^{p}\alpha_2 & + \cdots + & a_{p+1}^{p}\alpha_{p+1} = 1, \\
a_1^{p+1}\alpha_1 & + a_2^{p+1}\alpha_2 & + \cdots + & a_{p+1}^{p+1}\alpha_{p+1} = 0.
\end{array}$$

The set (26) is a set of $p+2$ non-homogeneous linear equations in the $(p+1)$ α's. Since the a's are distinct the (Vandermonde) determinant of coefficients of the first $(p+1)$ equations is not zero. From the elementary theory of linear equations it follows that (26) has a solution if and only if

$$\begin{vmatrix}
1 & 1 & \cdots & 1 & 0 \\
a_1 & a_2 & \cdots & a_{p+1} & 0 \\
a_1^2 & a_2^2 & \cdots & a_{p+1}^2 & 0 \\
\cdot & \cdot & \cdots & \cdot & \cdot \\
a_1^{p-1} & a_2^{p-1} & \cdots & a_{p+1}^{p-1} & 0 \\
a_1^{p} & a_2^{p} & \cdots & a_{p+1}^{p} & 1 \\
a_1^{p+1} & a_2^{p+1} & \cdots & a_{p+1}^{p+1} & 0
\end{vmatrix} = 0,$$

or simply $D=0$, where

$$(27) \qquad D = \begin{vmatrix}
1 & 1 & \cdots & 1 \\
a_1 & a_2 & \cdots & a_{p+1} \\
a_1^2 & a_2^2 & \cdots & a_{p+1}^2 \\
\cdot & \cdot & \cdots & \cdot \\
a_1^{p-1} & a_2^{p-1} & \cdots & a_{p+1}^{p-1} \\
a_1^{p+1} & a_2^{p+1} & \cdots & a_{p+1}^{p+1}
\end{vmatrix}$$

The determinant D is a product of all of the differences $(a_i - a_j)$, $i>j$, $i,j=1,2,\cdots,p+1$ by the sum $a_1+a_2+\cdots+a_{p+1}$ as the following argument shows. Evidently, D is a polynomial of degree P in the a's where

$$P = 0+1+2+\cdots+(p-1)+(p+1) = \frac{p(p+1)}{2}+1.$$

Since D vanishes if we set $a_i = a_j$ for any $i \neq j$, D contains $(a_i - a_j)$ as a factor for each pair of distinct values of i and j. There are thus $C(p+1,2)$ factors of D of the form $(a_i - a_j)$, $i<j$. Since

$$P - C(p+1,2) = 1,$$

D contains an additional factor of the first degree in the a's. Since D is homogeneous in the a's, this factor will be homogeneous in the a's, and is thus of the form $L = (k_1a_1 + k_2a_2 + \cdots + k_{p+1}a_{p+1})$. Thus D can be written as $D'L$, where D' is the product of the $(a_i - a_j)$, where $i>j$, and $i,j = 1,2,\cdots, p+1$. If we interchange two distinct elements a_i and a_j in D we obviously obtain from D a determinant equal to $-D$. In the same way this interchange changes the sign of D'. Hence this interchange does not affect L. Thus $k_1 = k_2 = \cdots = k_{p+1} = 1$. It follows that $D = 0$ if $a_1 + a_2 + \cdots + a_{p+1} = 0$. We choose distinct a's in K whose sum vanishes. Then (26) is solvable. Hence $m(x^py) \leqq p+1$. By Theorem 10 we have $m(x^py) > p$. Therefore $m(x^py) = p+1$.

Theorem 13. *For a field K of characteristic zero and forms F and L of degrees p and 1, respectively, $m(FL) \leqq (p+1)m(F)$.*

Now
$$F = \sum_{i=1}^{\rho} \lambda_i L_i^p,$$

whence
$$FL = \sum_{i=1}^{\rho} \lambda_i L L_i^p.$$

Since by Theorem 12, $m(LL_i^p) = p+1$, then FL has a representation with $\rho(p+1)$ terms, so that with $\rho = m(F)$ the theorem is proved.

Theorem 14. *If for a field of characteristic zero, a form F of degree p is a product of linear factors, then $m(F) \leqq p!$*

Suppose that $F = GL$ where L is linear. By Theorem 13, $m(GL) \leqq pm(G)$. If in turn $G = HK, K$ linear then $m(G) \leqq (p-1)m(H)$. Thus finally $m(F) \leqq p!$

Corollary. *If a quadratic form Q is a product of linear factors, the rank of Q is not greater than 2.*

Theorem 14 gives a necessary condition that a form F be a product of linear factors.

For the complex field Hocevar gave a necessary and sufficient condition for such factorability; his result is expressed in Theorem 15 below.

The *Hessian* of a form F in $x,y,\cdots, z$ is the determinant

$$\begin{vmatrix} \dfrac{\partial^2 F}{\partial x^2} & \dfrac{\partial^2 F}{\partial x \partial y} & \cdots & \dfrac{\partial^2 F}{\partial x \partial z} \\ \dfrac{\partial^2 F}{\partial y \partial x} & \dfrac{\partial^2 F}{\partial y^2} & \cdots & \dfrac{\partial^2 F}{\partial y \partial z} \\ \cdot & \cdot & \cdots & \cdot \\ \dfrac{\partial^2 F}{\partial z \partial x} & \dfrac{\partial^2 F}{\partial z^2 y} & \cdots & \dfrac{\partial^2 F}{\partial z^2} \end{vmatrix} .$$

Theorem 15. *A form F with no repeated factors is a product of linear factors if and only if F divides each third order determinant minor of its Hessian.*

For the proof of Theorem 15, the reader is referred to existing literature.

We have shown above how minimal numbers are related to factorization properties of forms. Necessary and sufficient conditions for the factorability of forms in certain ways can be expressed entirely in terms of minimal numbers and minimal representations. This treatment is beyond the scope of this paper, and will be omitted.

BIBLIOGRAPHY

I. L. E. Dickson, Modern Algebraic Theories.

II. Hočevar, Sur les formes decomposables en facteurs lineaires, Comptes Rendus de l'Academie des Sciences, Vol. 138 (1904), pp. 745-747.

III. R. Oldenburger, Composition and rank of n-way matrices and multilinear forms, Annals of Mathematics, Vol. 35 (1934), pp. 622-657.

IV. R. Oldenburger, Representation and equivalence of forms, Proceedings of the National Academy of Sciences, Vol. 24 (1938), pp. 193-198.

V. B. L. Vander Wærden, Moderne Algebra.

VI. H. Weyl, Gruppentheorie und Quantenmechanik.

Humanism and History of Mathematics

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A History of American Mathematical Journals

By BENJAMIN F. FINKEL
Drury College

(Continued from May, 1940, issue)

As has been said, the most important contribution to the *Diary* is the article on *Perfect Numbers* by Benjamin Pierce. Because of the scarcity of the *Diary*, this article is unknown to even some specialists in Number Theory. It may therefore be well to reproduce this article in full.

ON PERFECT NUMBERS

By Benjamin Pierce, Esq., Mathematical Instructor in Harvard University, Cambridge, Massachusetts.

Definition. A Perfect Number is one that is equal to the sum of all its factors, unity, included.

Euler demonstrated that $(2^{n+1}-1)2n$ is a perfect number when $2^{n+1}-1$ is a prime number.* But I have never seen it satisfactorily demonstrated, that this form includes all perfect numbers; Barlow's attempt is plainly defective. It is my object, in the present paper, to show that there can be no other perfect number included in the forms, a^m, $a^m b^n$, $a^m b^n c^p$, where a, b, and c are prime numbers and greater than unity.

Notation.—Represent by $\Sigma^m a^i$ the sum of all the powers of a, from zero to the mth power inclusive, that is, let

$$\Sigma^m a^i = 1 + a + a^2 + a^3 + \cdots + a^m.$$

Let wh stand for the word integer, or whole number.

* So did Euclid.

FIRST FORM, a^m

The sum of the factors of a^m is

$$1+a+a^2+a^3+\cdots+a^{m-1}a^t=\Sigma^{m-1}a^t.$$

But by the definition of a^m, this sum is equal to a^m, or $\Sigma^{m-1}a^t=a^m$. Every term of each member of this equation is divisible by a, except the term 1. Therefore, a cannot be greater than 1, and no perfect number can be included in this first form.

SECOND FORM, $a^m b^n$.

The sum of all the factors of $a^m b^n$ is,

$$\begin{aligned}&1+a+b+a^2+ab+b^2\cdots\cdots+a^{m-1}b^n+a^m b^{n-1}\\&=1+a+b+a^2+ab+b^2+\cdots\cdots+a^m b^n-a^m b^n,\\&=(1+a+a^2+\cdots\cdots+a^m)(1+b+b^2+\cdots\cdots+b^n)-a^m b^n\\&=\Sigma^m a^t\times\Sigma^n b^t-a^m b^n.\end{aligned}$$

But since $a^m b^n$ is a perfect number, this sum must equal $a^m b^n$, or

$$\Sigma^m a^t \Sigma^n b^t-a^m b^n=a^m b^n,$$

$$\Sigma^m a^t\times\Sigma^n b^t=2a^m b^n.$$

So that $\Sigma^m a^t$ or $\Sigma^n b^t$ must be an even number, and as $\Sigma^m a^t$ cannot be divisible by a and $\Sigma^n b^t$ cannot be divisible by b. We may suppose

$$\Sigma^m a^t=2b^n, \text{ and } \Sigma^n b^t=a^m.$$

Hence, $\Sigma^n b^t-1=a^m-1$, and

$$\frac{a^m-1}{b}=\frac{\Sigma^n b^t-1}{b}=wh.$$

But we had $$2b^n=\Sigma^m a^t=\frac{a^{m+1}-1}{a-1};$$

therefore, $$\frac{a^{m+1}-1}{b}=wh.$$

So that $$\frac{a^{m+1}-1}{b}-\frac{a(a^m-1)}{b}=\frac{a-1}{b}=wh.=p \text{ and } >0;$$

and $$a=pb+1.$$

Again $\qquad \Sigma^m a^t - 1 = 2b^n - 1,$

and $$\frac{2b^n - 1}{a} = \frac{\Sigma^m a^t - 1}{a} = wh.$$

But $$a^m = \Sigma^n b^t = \frac{b^{n+1} - 1}{b-1};$$

so that $$\frac{b^{n+1} - 1}{a} = wh.$$

Or $$2\left(\frac{b^{n+1} - 1}{a}\right) - b\left(\frac{2b^n - 2}{a}\right) = \frac{b-2}{a} = \frac{b-2}{pb+1},$$

which is impossible, unless $b-2=0$, or $b=2$.

This gives $$\Sigma^n b^t = \frac{b^{n+1} - 1}{b-1} = 2^{n+1} - 1 = a^m,$$

and $\qquad \Sigma^m a^t = 2b^n = 2^{n+1};$

hence, $\qquad \Sigma^m a^t - a^m = \Sigma^{m-1} a^t = 1.$

So that $\qquad m-1=0, \text{ or } m=1,$

and $\qquad a = 2^{n+1} - 1.$

Therefore, $(2^{n+1}-1)2^n$ is a perfect number, when $2^{n+1}-1$ is a prime number. This is the form given by Euler, and it is evidently a perfect number, for the sum of its factors is

$$(1+2+2^2+\cdots+2^n)+(1+2+2^2+\cdots+2^{n-1})(2^{n+1}-1)$$
$$=(2^{n+1}-1)+(2^n-1)(2^{n+1}-1)=2^n(2^{n+1}-1) \text{ as required.}$$

THIRD FORM, $a^m b^n c^p$.

The sum of the factors of $a^m b^n c^p$, is found as in the former case to be $\Sigma^m a^t \times \Sigma^n b^t \times \Sigma^p c^t - a^m b^n c^p = a^m b^n c^p$, or $\Sigma^m a^t \times \Sigma^n b^t \times \Sigma^p c^t = 2a^m b^n c^p$.

Whence we may suppose, as before,

$$\Sigma^m a^t = 2b^{n'} c^{p'}$$
$$\Sigma^n b^t = a^{m''} c^{p''}$$
$$\Sigma^p c^t = a^{m'''} b^{n'''},$$

m'', m''', n', n''', p', and p'' being integers, or equal to zero.

And satisfying the equations,

$$m''+m'''=m,$$

$$n'+n'''=n,$$

$$p'+p''=p;$$

it follows from $\Sigma^m a^t = 2b^{n'}c^{p'}$, that m and a are both of them odd numbers.

CASE FIRST.

Suppose $b>2$ and $c>2$. Then they must both be odd numbers, and n and p must both be even numbers.

1. *m cannot be greater than* 1. For suppose $m>1$, then

$$\Sigma^m a^t = \frac{a^{m+1}-1}{a-1} = (a^{(m+1)/2}+1)\left(\frac{a^{(m+1)/2}-1}{a-1}\right) = 2b^{n'}c^{p'}.$$

Now
$$\frac{a^{(m+1)/2}+1}{b} \text{ or } \frac{a^{(m+1)/2}+1}{c}$$

is an integer, we may suppose the former.

Then
$$\frac{a^{(m+1)/2}+1}{b} = wh. \text{ and } \frac{(a^{(m+1)/2}+1)-2}{b} = \frac{a^{(m+1)/2}-1}{b} = wh.$$

is impossible. We must, therefore, have

$$\frac{a^{(m+1)/2}-1}{c} = wh. \text{ and } \frac{(a^{(m+1)/2}-1)+2}{c} = \frac{a^{(m+1)/2}+1}{c} = wh.$$

is impossible. Moreover,

$$\frac{a^{(m+1)/2}-1}{a-1} = wh. \text{ and } a^{(m+1)/2}+1$$

is an even number; we must, therefore, have $a^{(m+1)/2}+1 = 2b^{n'}$, and

$$\frac{a^{(m+1)/2}-1}{a-1} = c^{p'} = \Sigma^{(m+1)/2-1}a^t.$$

From this last, it is plain that

$$\frac{m+1}{2} - 1$$

is an even number, and, therefore, $\frac{m+1}{2}$ is an odd number, and $a+1$ is a factor of $a^{(m+1)/2}+1=2b^{n'}$. Therefore, $a+1=2b^d$, d being any positive integer, and $a=2b^d-1$, $2b^{n'}=(2b^d-1)^{(m+1)/2}+1$. The term containing b^d is $2\left(\frac{m+1}{2}\right)b^d$, which must be divisible by $2b^{d+1}$, unless $d=n'$, which cannot be, since it would give

$$2b^{n'}-1=2b^d-1=(2b^d-1)^{(m+1)/2},$$

$$\frac{m+1}{2}=1,\quad m=1,$$

contrary to the present hypothesis.

We must then have $\frac{m+1}{2}=eb^g$, e being prime to b, and g an integer.

Hence, $$2b^{n'}=(2b^d-1)^{eb^g}+1;$$

therefore, $(2b^d-1)^{b^g}+1$ being a factor of $(2b^d-1)^{eb^g}+1$, we may write $(2b^d-1)b^g+1=2b^h$; or $(2b^d-1)b^g-2b^h+1=0$, or, developing the last terms, and D being a known function of d,

$$D\cdot b^{3d}-2b^g(b^g-1)b^{2d}+2b^db^g-2b^h=0,$$

$$D\cdot b^{3d}-2b^{2g+2d}+2b^{2d+g}+2b^{d+g}-2b^h=0,$$

which evidently requires that $d+g=h$, and this being substituted, the equations becomes $Db^{3d}-2b^{2g+2d}+2b^{2d+g}=0$,

or $$D'\cdot b^{4d}+4b^g(b^g-1)\left(\frac{b^g-2}{3}\right)b^{3d}-2b^{2g+2d}+2b^{2d+g}=0,$$

which can be satisfied only by making $b=3$, and then it becomes

$$D''\cdot 3^{g+3}+4\cdot 3^{3g+3d-1}-4\cdot 3^{2g+3d}-2\cdot 3^{2g+2d}+8\cdot 3^{3d+g-1}+2\cdot 3^{2d+g}=0;$$

so that $3d+g-1=2d+g$ or $d=1$, and the equation becomes

$$D''\cdot 3^{g+3}+4\cdot 3^{3g+2}-4\cdot 3^{2g+3}-2\cdot 3^{2g+2}+8\cdot 3^{g+2}=0,$$

which is plainly impossible. Therefore m cannot exceed unity.

This fact reduces the equations for case first to these:

$$1+a=2b^{n'}c^{p'},$$

$$\Sigma^n b^i=ac^{p''},$$

$$\Sigma^p c^i=b^{n'''};$$

for $m''+m'''=m=1$. Therefore $m''=0$ or $m'''=0$, and it is a matter of indifference which is made zero.

These equations give $$a=2b^{n'}c^{p'}-1,$$

$$\Sigma^n b^i=2b^{n'}c^p-c^{p''},$$

$$\Sigma^p c^i=\frac{c^{p+1}-1}{c-1}=b^{n'''}.$$

Hence, $$\frac{c^{p''}+1}{b}=wh. \text{ unless } n'=0, \text{ and } \frac{c^{p+1}-1}{b}=wh.$$

Also $$\frac{c^{p''(p+1)}+1}{c^{p''}+1} \text{ being an integer, } \frac{c^{p''(p+1)}+1}{b}=wh.$$

and $$\frac{c^{p''(p+1)}-1}{c^{p+1}-1} \text{ being an integer, } \frac{c^{p''(p+1)}-1}{b}=wh.$$

so that $$\frac{c^{p''(p+1)}+1}{b}+\frac{c^{p''(p+1)}-1}{b}=\frac{2}{b}=wh.$$

which is impossible.

Therefore, $$n'=0, \quad \text{and}$$

$$a=2c^{p'}-1,$$

$$\Sigma^n\cdot b^i=2c^p-c^{p''}=a\cdot c^{p''}=\frac{b^{n+1}-1}{b-1}$$

$$\Sigma^p c^i=b^n \quad .$$

Hence, $$b^{n+1}-1=ac^{p''}(b-1)=b\Sigma\cdot{}^p c^i-1,$$

$$b=\frac{ac^{p''}-1}{ac^{p''}+\Sigma\cdot{}^p c^i}=\frac{ac^{p''}-1}{A}=wh.$$

where $A=ac^{p''}-\Sigma\cdot{}^p c^i$.

Hence, $$(ac^{p''}-1-A)\div A=\frac{\Sigma\cdot{}^p c^i-1}{A}=c\left(\frac{\Sigma^{p-1}c^i}{A}\right)=wh.$$

or $$\frac{\Sigma^{p-1}\cdot c^i}{A}=wh. \text{ and } \frac{c^p-1}{A}=wh.$$

Also $$\frac{ac^{p''}-1}{A}-\frac{c^p-1}{A}=\frac{ac^{p''}-c^p}{A}=c^{p''}\left(\frac{a-c^{p'}}{A}\right)=wh.$$

or $$\frac{a-c^{p'}}{A}=\frac{c^{p'}-1}{A}=wh.$$

Let the greatest common divisor of p and p' be p_1; x and y can be found such that $xp-y^{p'}=p_1$,

then $$\frac{c^{xp}-1}{A}=wh. \text{ and } \frac{c^{yp'}-1}{A}=wh.$$

therefore,

$$\frac{c^{xp}-1}{A}-\frac{c^{yp'}-1}{A}=\frac{c^{xp}-c^{yp'}}{A}=c^{yp'}\left(\frac{(c^d-1)}{A}\right)=wh.$$

and $$\frac{c^d-1}{A}=wh.=\frac{c^{d-1}}{c^p-c^{p''}-\Sigma^{p-1}\cdot c^i}$$

$$=\frac{c^{d-1}(c-1)}{(c-1)c^{p'}\cdots\cdot 1(c^{p''}\cdots\cdots(c^p\cdots\cdot 1)}$$

or $$(c-1)\div\left((c-1)c^{p''}\frac{c^{p'}-1}{c^d-1}-\frac{c^p-1}{c^d-1)}\right)=wh.$$

$$=(c-1)\div(Pc-1),$$

where P is an integer, unless $p''=0$, which would give $\Sigma^n b^i=2c^p-1$, or $2/b=wh.$ which is impossible.

Therefore, $$\frac{c-1}{Pc-1}=wh. \quad \text{or} \quad P=1,$$

for $P=0$ would make A and, therefore, b negative quantities.

Hence, $$(c-1)c^{p''}\left(\frac{c^{p'}-1}{c^d-1}\right)-\frac{c^p-1}{c^d-1}=c-1,$$

and by development

$$c^{p+1}-2c^p-c^{p''+1}+c^{p''}-c^{d+1}+c^d+c=0,$$

or $$c^p-2c^{p-1}-c^{p''}+c^{p''-1}-c^d+c^{d-1}+1=0$$

which requires that $d=1$.

Then $$c^p-2c^{p-1}-c^{p''}+c^{p''-1}-c+2=0;$$

therefore, $p''=1$, $\quad c^p-2c^{p-1}-2c+3=0.$

Therefore, $c=3$, for p being even cannot equal unity.

This gives $3^p-2\cdot 3^{p-1}-3=0$, or $3^{p-1}-2\cdot 3^{p-2}-1=0$.

Hence, $p=2$, and $a=2c^{p'}-1=5$,

$$\Sigma^p\cdot c^i=1+3+3^2=13=b^n=b,$$

$$\Sigma^n\cdot b^i=1+13=ac^{p''}=15, \quad \text{which is absurd.}$$

It follows from the preceding analysis, that b and c cannot be both different from 2.

[EDITORS NOTE: The Second Case, namely, when $b=2$, will be considered in the opening of the November installment of this History.]

"Each of five men—Lobachewsky, Bolyai, Plücker, Riemann, Lie—invented as part of his lifework as much (or more) new geometry as was created by all the Greek mathematicians in the two or three centuries of their greatest activity. There are good grounds for the frequent assertion that the nineteenth century alone contributed about five times as much to mathematics as had all preceding history. This applies not only to quantity but, what is of incomparably greater importance, to power."—*The Development of Mathematics* by E. T. Bell, New York, 1940, p. 15.

The Teacher's Department

Edited by
JOSEPH SEIDLIN and JAMES MCGIFFERT

Insights or Trick Methods?

A. L. O'TOOLE
Mundelein College, Chicago, Illinois

Judd* has said that the purpose of higher education is to give students insights. Insight adds a feature to an experience that trial and error cannot give alone. It is probably safe to say that most teachers and textbook writers agree that generalization is the very soul of mathematics. But then I'm afraid that most of us proceed to teach certain sections of elementary mathematics in a way that discourages students by giving them the inpression that excellence in mathematical science is a matter of trick methods and even legerdemain. The usual College Algebra treatment of the summing of series may be an example of this.

By a special trick that works only for an Arithmetic Progression the student is shown that the sum of the first n terms of an A. P. is $S_n = (n/2)(a+L)$. By another special trick that works only for a Geometric Progression he is shown that the sum of the first n terms of a G. P. is $S_n = a(1-r^n)/(1-r)$. The suggestion of all this to the student is that he must go on discovering tricks, one for each new series that he meets. The student who has already become a fatalist with respect to mathematical science will accept this suggestion as a matter of course. But some of the students may be disappointed with such a state of affairs and ask if there is some general way to attack the summing of series. Many times the answer given to these students is that the discussion of other methods of summing series "is beyond the scope of the course". Perhaps we are often too ready with that reply. In the case of series there is a general method of summing and, moreover, there is nothing involved in the method that a student of College Algebra cannot understand. No doubt the good student will be encouraged and stimulated more by one general method that applies

*Charles Hubbard Judd, The Psychology of Secondary Education. Boston: Ginn and Company, 1927.

to innumerable problems than by any number of special tricks each of which applies to only one problem.

The fundamental theorem of summation states that if there is a function $f(x)$ such that $u_x = f(x+1) - f(x)$, then

$$\sum_{x=a}^{b} u_x = f(b+1) - f(a) = f(x)\Big]_a^{b+1}.$$

The proof of the theorem is extremely simple. For by hypothesis we have:

$$\begin{aligned}
u_a &= f(a+1) - f(a) \\
u_{a+1} &= f(a+2) - f(a+1) \\
u_{a+2} &= f(a+3) - f(a+2) \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
u_{b-1} &= f(b) - f(b-1) \\
u_b &= f(b+1) - f(b)
\end{aligned}$$

Summing, $$\sum_{x=a}^{b} u_x = f(b+1) - f(a). \qquad \text{Q. E. D.}$$

Any College Algebra student can apply the theorem to the summation of series such as:

$$a,\ a+d,\ a+2d, \cdots\cdots,\ a+(n-1)d$$

$$a,\ ar,\ ar^2, \cdots\cdots\cdots,\ ar^{n-1}$$

$$1^p,\ 2^p,\ 3^p,\ \cdots\cdots\cdots,\ n^p$$

$$\frac{r}{1\cdot 2\cdot 3},\ \frac{r^2}{2\cdot 3\cdot 4}, \cdots\cdots\ \frac{r^x}{x(x+1)(x+2)}$$

Before proceeding to illustrate the application of the theorem it may be well to call attention to the two most difficult steps in the process. The first is writing the general term of the series. The second is determining the nature of the function $f(x)$. One of the purposes of the Calculus of Finite Differences is to develop techniques for dealing with such function problems. For many problems all that the student needs is some experience with functional notation and some practice in determining $f(x+1) - f(x)$ for simple functions such as polynomials and exponentials. For instance, the student should not have much difficulty in realizing that if $f(x+1) - f(x)$ is a polynomial

of the second degree $f(x)$ must be a polynomial of the third degree. Illustrations:

(1) $$S_n = a+(a+d)+(a+2d)+\cdots\cdots+[a+(n-1)d] = \sum_{x=0}^{n-1}(a+xd).$$

Since $u_x = a+xd$ is a polynomial of the first degree then $f(x)$ is a polynomial of the second degree. Let $f(x)=Ax^2+Bx+C$. Then

$$f(x+1)-f(x)=2Ax+(A+B)=a+xd$$

and, equating coefficients, $A=d/2$ and $B=a-d/2$. Thus C is arbitrary. Hence, applying the theorem,

$$S_n=\left[\frac{d}{2}x^2+\left(a-\frac{d}{2}\right)x+C\right]_0^n=\frac{n}{2}[2a+(n-1)d].$$

(2) $$S_n=a+ar+ar^2+\cdots\cdots\cdots\cdots+ar^{n-1}=\sum_{x=0}^{n-1}ar^x\ .$$

Since $u_x=ar^x$ it will be clear after the practice suggested above that for this case $f(x)=kr^x$ where k is to be determined. Then $f(x+1)-f(x)$ $=kr^{x+1}-kr^x=k(r-1)r^x=ar^x$ and hence $k=a/(r-1)$. Therefore

$$S_n=\left[\frac{ar^x}{r-1}\right]_0^n = \frac{a(r^n-1)}{r-1}\ .$$

(3) $$S_n=1^2+2^2+3^2+\cdots\cdots\cdots\cdots\cdots+n^2=\sum_{x=1}^{n}x^2\ .$$

Here $u_x=x^2$ which is a second degree polynomial.

Hence let $$f(x)=Ax^3+Bx^2+Cx+D.$$

Then $$f(x+1)-f(x)=3Ax^2+(3A+2B)x+(A+B+C)=x^2.$$

Therefore $A=1/3$, $B=-1/2$, $C=1/6$ and D is arbitrary. Let $D=0$.

Then

$$S_n=\left[\frac{2x^3-3x^2+x}{6}\right]_1^{n+1}=\frac{n(n+1)(2n+1)}{6}\ .$$

(4) $$S_n=\frac{r}{1\cdot2\cdot3}+\frac{r^2}{2\cdot3\cdot4}+\cdots\cdots+\frac{r^n}{n(n+1)(n+2)}$$

$$=\sum_{x=1}^{n}\frac{r^x}{x(x+1)(x+2)}\ .$$

Experience will suggest here that one should let $f(x)$ be of the form

$$\frac{r^x F(x)}{x(x+1)}$$

where $F(x)$ is a rational, integral function. Then

$$f(x+1)-f(x)=\frac{xr^{x+1}F(x+1)-(x+2)r^xF(x)}{x(x+1)(x+2)}=\frac{r^x}{x(x+1)(x+2)}$$

and therefore $xrF(x+1)-(x+2)F(x)=1$ and $F(x)=c$, where c is a constant different from zero.

$$cxr-c(x+2)=1$$

$$(cr-c)x-2c=1.$$

Equating coefficients,

$$c(r-1)=0 \text{ and } -2c=1.$$

Hence the series cannot be summed unless $r=1$. With $r=1$ and $c=-1/2$

$$S_n=\left.\frac{-1}{2x(x+1)}\right]_1^{n+1}=\frac{1}{4}-\frac{1}{2(n+1)(n+2)}\;.$$

To say that such a treatment of series in College Algebra would enable the student to sum every summable series would be ridiculous But the summing of series which he will meet in more advanced courses will be only an amplification of what he has learned previoulsy. Formulas such as the powerful Euler-Maclaurin Sum Formula are based on the fundamental theorem of summation stated above. Having given the student the confidence and insight that he will get from this theorem it may be quite proper then to discuss with him the tricks that work for the A. P. and the G. P. Some English books give tricks for other series too. Or, if it is felt that there is time for the tricks only, at least a reference can be given to the student so that he will know that there is a "general method."

Problem Department

Edited by
ROBERT C. YATES and EMORY P. STARKE

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscript be typewritten with double spacing. Send all communications to ROBERT C. YATES, Mathematics, University, Louisiana.

NOTICE

In order to promote more active participation in this department on the part of young readers, the editors sponsor the following contest. For the best printed Proposal and for the best printed Solution of any problem, awards will be made of a year's subscription to this Magazine. The contest is open to graduate and undergraduate students only. Winners will be announced in October, 1941.

SOLUTIONS

Late Solutions: No. 339 by *C. D. Smith, Jr.*, No. 341 by *George Yanosik.*

No. 321. Proposed by *A. A. Aucoin* and *W. V. Parker*, Louisiana State University.

Determine a five-parameter solution for

$$(x+2y+z)(x+y)(y+z)(x+2z)=u^2+2uv+5v^2.$$

Solution by *A. A. Aucoin*, University of Houston.

To solve

$$(x+2y+z)(x+y)(y+z)(x+2z)=u^2+2uv+5v^2, \tag{1}$$

let

$$\left\{\begin{aligned} x+2y+z&=-3at\\ x+y&=-3bt\\ x+2z&=-3cs\end{aligned}\right. \tag{2} \qquad \text{and (3)} \left\{\begin{aligned} u&=3^2mst\\ v&=3^2nst.\end{aligned}\right.$$

Solving (2) we get

$$x=(2a-4b)t-cs, \quad y=(-2a+b)t+cs, \quad z=(-a+2b)t-cs \tag{4}$$

and thus $y+z=-3(a-b)t$. With these results (1) becomes

$$3^4abc(a-b)st^3=3^4(m^2+2mn+5n^2)s^2t^2 \quad \text{or}$$

$$s/t=abc(a-b)/(m^2+2mn+5n^2). \tag{5}$$

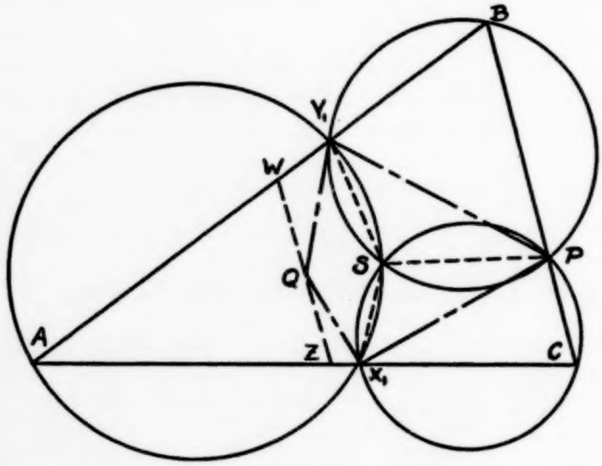
Hence a solution of (1) in terms of the parameters a, b, c, m, n, is given by (3) and (4), where s and t are chosen according to (5)*

Furthermore, every solution of (1) is given as above. Let x, y, z, u, v be numbers satisfying (1). Corresponding values of a, b, c, m, n, for arbitrary s and t, may be determined from the linear equations (2) and (3). Values so determined will satisfy (5) which is merely the result of eliminating x, y, z, u, v from (2), (3) and (1)

No. 324. Proposed by *Walter B. Clarke*, San Jose, California.

Given a point P and two intersecting lines, all in a plane. (1) Construct two equal and perpendicular line segments from P to the given lines. (2) Let Q be the vertex that lies opposite P of a square on the segments as adjacent sides. What is the locus of Q as P describes a line?

Solution by *C. D. Smith*, Mississippi State College.



*Different values for s and t (say ks and kt), corresponding to the same a, b, c, m, n, only multiply x, y, z by k and u, v by k^2. Thus the two resulting solutions are not independent.

The solution is a special case of a more general problem which is given by the following Three-circle construction. With triangle ABC as triangle of reference the point S is Miquel point of the triangle PX_1Y_1. As the vertices P, X_1, and Y_1 move along the respective sides of ABC so that the fixed point S is the Miquel point of each triangle, point S is the center of similitude of the set of triangles and the angles at S are supplements of angles at the respectively opposite vertices of ABC in each circle. When S falls on P in the illustration so that P is fixed while X_1 moves on AC and Y_1 moves on AB then circle AX_1SY_1 becomes the circumcircle of PX_1Y_1 and the locus of Y_1 is AY_1B. Part 1 is the special case where triangle PX_1Y_1 is isosceles with angle P a right angle.

Obviously if S is a fixed point different from P, a fourth point Q is sufficient to determine circles QSX_1, and QSY_1, which determine Z on AC and W on AB such that ZW is the locus of Q where the set of triangles QX_1Y_1 have S for Miquel point and triangle AZW for triangle of reference. The set of similar quadrilaterals PX_1QY_1 now have S as center of similitude and ZW is the locus of Q when BC is the locus of P. Part 2 is the special case where we begin with $PX_1=PY_1$ and angle P a right angle so that the quadrilateral PX_1QY_1 is a square. (See Johnson's *Modern Geometry*, Chap. VII and a recent paper by the solver: *Three-Circle Problems in Modern Geometry*, this Magazine, XIV, 1940, pp. 299-307).

Also solved by *D. L. MacKay*.

No. 333. Proposed by *Paul D. Thomas*, Norman, Oklahoma.

Prove that the locus of the feet of the perpendiculars drawn from the vertex of a triangle upon the polars of that vertex with respect to the circles of the coaxal pencil determined by the other two vertices is an Apollonian circle.

Solution by the *Proposer*.

Let the vertices of a triangle be $P_1(f,g)$, $P_2(0,c^{\frac{1}{2}})$, $P_3(0,-c^{\frac{1}{2}})$. The equation of the coaxal pencil through P_2 and P_3 is

(1) $$x^2+y^2+2ax-c=0. \quad (a \text{ is a parameter}).$$

The polar of P_1 with respect to (1) is (2): $(a+f)x+gy+af-c=0$. The line through P_1 perpendicular to (2) is (3): $gx-(a+f)y+ag=0$. The locus of the intersection of (2) and (3) is found by eliminating the parameter a between these two equations.

From (3), $a=(fy-gx)/(g-y)$. This value of a placed in (2) produces the circle (4): $g(x^2+y^2)-(g^2+f^2+c)y+cg=0$, whose center is

$[0,(g^2+f^2+c)/2g]$ and radius $[(g^2+f^2)^2+2c(f^2-g^2)+c^2]^{1/2}/2g$. (4) is an Apollonian circle if it has for diameter the segment on P_2P_3 intercepted by the internal and external bisectors of angle $P_2P_1P_3$. The equations to these bisectors are

$$[(g-c^{1/2})x-fy+fc^{1/2}]/[(g-c^{1/2})^2+f^2]^{1/2}$$
$$= \pm[(g+c^{1/2})x-fy-fc^{1/2}]/[(g+c^{1/2})^2+f^2]^{1/2}.$$

These bisectors meet P_2P_3 in the points $S[0,c^{1/2}(B-A)/(A+B)]$ and $T[0,c^{1/2}(A+B)/(B-A)]$, where $A=[(g-c^{1/2})^2+f^2]^{1/2}$, $B=[(g+c^{1/2})^2+f^2]^{1/2}$. The midpoint of the segment ST is $[0,c^{1/2}(A^2+B^2)/(B^2-A^2)]$ or, replacing A and B by their values, $[0,(g^2+f^2+c)/2g]$ which is the center of circle (4). The length of the segment ST is

$$4c^{1/2}AB/(B^2-A^2)=[(g^2+f^2)^2+2c(f^2-g^2)+c^2]^{1/2}/g$$

which is the diameter of (4). Hence (4) is an Apollonian circle of triangle $P_1P_2P_3$.

Also solved by *Johannes Mahrenholz.*

No. 340. Proposed by *V. Thébault,* Le Mans, France.

Find such a four-digit number that when 385604 is written at its right the result is a perfect square.

Solution by *George A. Yanosik,* New York University.

We require $N^2=\cdots 385604$ with $31622<N<100000$. Noting that $98^2=9604$, which agrees with N^2 in the last three digits, we must have $N=500a\pm 98$. If $N^2=(500a\pm 98)^2=\cdots 5604$, we must have $N=2500b\pm 1098$.* For N^2 to terminate in 85604, we need

$$N=12500c\pm 6098,$$

with $3\leqq c\leqq 8$ to insure the inequality on N. Then we have

$$N^2=(12500c\pm 6098)^2=156250000c^2\pm 152450000c+37185604$$
$$=\cdots 385604, \text{ or } 15625c^2\pm 15245c=\cdots 20.$$

This is satisfied (for c within the above limits) only when $c=4$ and the lower sign is taken. Thus N is 43902, and 1927 is the required four-digit number to make $43902^2=1927385604$.

Also solved by *C. W. Trigg.*

*Because $(500a\pm 98)^2=250000a^2\pm 98000a+9604=\cdots 385604$ or $125a^2\pm 49a=\cdots 8$ requires $a=5b\pm 2$, whence $N=2500b\pm 1098$.—ED.

No. 342. Proposed by *Althéod Tremblay*, Québec, Canada.

A well known problem in algebra is as follows: "An American city has m streets running east and west and n avenues running north and south. In how many ways can a man go from the southwest corner to the northeast corner if he always faces either north or east?" The answer is $(m+n)!/(m!\cdot n!)$.

This problem suggests the following question. If the man of the problem chooses his path at random, what is the probability of his meeting a friend who is waiting at the corner of Avenue P and Street Q?

Solution by the *Proposer*.

Suppose first an unlimited number of avenues and streets. At each corner our man can choose two ways, north or east, and either of these choices is equally likely. Further, to reach any corner, C, he must previously have passed through either A, the next corner to the west, or B, the next to the south. Thus the probability of reaching C through A is half the probability of passing through A, and the probability of reaching C is half the sum of the probabilities for A and B. If C is the intersection of the pth avenue and the qth street, an easy induction establishes the probability of meeting his friend at C to be

$$(p+q-2)!/(p-1)!(q-1)!2^{p+q-2}.$$

If now the number of avenues and streets is limited, the above formula holds for all corners except those on the border avenue and on the border street. Thus for the three corners mentioned above, if C is on the easternmost avenue, the probability of reaching C is half the probability for A plus the probability for B, since at B there is no way to go except north. Similarly for corners on the northernmost street.

To illustrate, suppose there are seven avenues and four streets. The above discussion leads to the probabilities of passing through the various corners as shown at the corresponding intersections in the following diagram. Each probability shown has been multiplied by 2^8.

32	80	128	163	198	219	256
64	96	96	80	60	42	37
128	128	96	64	40	24	16
256	128	64	32	16	8	4

$(\div 256)$

Note that the probabilities along any northwest-southeast diagonal are the terms of $(\frac{1}{2}+\frac{1}{2})^k$. When the diagonal is cut short by 7th avenue or 4th street, the probability shown there is the sum of all the remaining terms of $(\frac{1}{2}+\frac{1}{2})^k$.

Editor's Note. The answer given in the first paragraph of the proposal should read $(m+n-2)!/(m-1)!(n-1)!$ It would have been correct for a rectangle of m by n blocks, having thus $m+1$ streets and $n+1$ avenues.

The Proposer's solution given above rests on a very delicate point in the theory of probability. It will be readily admitted that when the number of streets and avenues is unlimited, the man is *equally likely* to go either east or north. When, however, m and n are finite this is subject to doubt. Thus, if the man has yet to go five blocks east and one north, is he not more likely to choose east? (A familiar analogous argument: since at the end of the next 24 hours I must be either living or dead, is the probability of death $\frac{1}{2}$?) One might ask, referring to the diagram, why was the man not equally likely to reach any corner *from* the west or *from* the south? On the other hand, the words, "chooses $\cdots$ at random", of the proposal may perhaps be interpreted to allow the above.

A simpler and more satisfying solution is based on the agreement: the probability of passing through corner C is the ratio of the number of possible paths which go through that corner to the total number of paths. Then the number of paths through C is the product of the number of ways of reaching C from the southwest corner of town by the number of ways of going from C to the northeast corner. If C is the intersection of the pth avenue and the qth street, we have as our desired probability:

$$\frac{(p+q-2)!}{(p-1)!(q-1)!}\cdot\frac{(n-p+m-q-2)!}{(n-p-1)!(m-q-1)!}\div\frac{(m+n-2)!}{(m-1)!(q-1)!}\,.$$

For comparison, the probabilities in the particular case cited by the proposer will now appear as follows, in which each probability has been multiplied by 84:

1	4	10	20	35	56	84	
7	18	30	40	45	42	28	
28	42	45	40	30	18	7	$(\div 84)$.
84	56	35	20	10	4	1	

PROPOSALS

No. 366. Proposed by *Paul D. Thomas*, Norman, Oklahoma.

The polar of the vertex A with respect to the variable circle through the vertices B and C of triangle ABC meets the circle in the points P and Q. The perpendicular from A upon this polar meets the circle in the points R and S.

1. P and Q trace the cubic curve T_1.
2. R and S trace the cubic curve T_2 which is orthogonal to T_1 at the vertices of ABC.
3. T_1 is tangent to the circumcircle of ABC at A.
   T_2 is tangent at A to the Appolonian circle of ABC passing through A.
4. The center of curvature of T_1 at A is the midpoint of the segment joining the circumcenter to A.
5. The center of curvature of T_2 at A is the midpoint of the exsymmedian of ABC issued from A.

No. 367. Proposed by *Dewey C. Duncan*, Los Angeles City College.

Find all rectangular parallelopipeds whose principal diagonals are 49 units in length and whose edges are all of integral lengths.

No. 368. Proposed by *F. C. Gentry*, Louisiana Polytechnic Institute.

If A, B, C and a, b, c are respectively the angles and the lengths of the opposite sides of a triangle then

$$\begin{vmatrix} a\cos^2 A & b\cos^2 B & c\cos^2 C \\ \cos A & \cos B & \cos C \\ a & b & c \end{vmatrix} = 0.$$

No. 369. Proposed by *Dewey C. Duncan*, Los Angeles City College.

Let a, b, c, d be integers with no common factor such that

$$a^2+b^2+c^2=d^2.$$

Prove:

(i) d is odd; of a, b, c, two are even and one odd.
(ii) 12 is the greatest number which always divides $abcd$.
(iii) if 3 divides d, 3 does not divide abc.
(iv) if 3 does not divide d, 9 divides abc.
(v) if $d=4k-1$, 8 does not divide abc.
(vi) if $d=4k+1$, 16 divides abc.

No. 370. Proposed by *N. A. Court,* University of Oklahoma.

The vertices of a variable tetrahedron with a fixed circumcenter lie on two fixed skew lines. (1) Show that the centroids of the faces lie on two fixed straight lines. (2) May the tetrahedron in some of its positions become isosceles (i. e., may each edge become equal to the respectively opposite edge)?

No. 371. Proposed by *C. C. Chaudoir,* Baker, Louisiana.

Prove: In a right triangle whose sides are integers without common divisor, if a leg B is a power of 2 then the other leg and the perimeter are each divisible by $\frac{1}{2}(B+2)$.

No. 372. Proposed by *D. L. MacKay,* Evander Childs High School, New York.

Show that it is impossible with straightedge and compasses to construct two lines perpendicular to each other which will quadrisect a given triangle ABC.

No. 373. Proposed by *V. Thébault,* Le Mans, France.

Find the smallest possible base of a system of numeration in which the three-digit number 777 is a perfect fourth power.

More than 500 members of three mathematical organizations, the American Mathematical Society, the Mathematical Association of America, and the Algebra Conference, will attend meetings at the University of Chicago in 1941, held in connection with the celebration of the University's fiftieth anniversary.

The three mathematical groups are among thirty-one learned societies, more than 15,000 of whose members will gather at the University in its Anniversary Year, which begins in October, 1940, culminating in an academic festival in September, 1941.

Announcement at the University of the meetings was made by Frederic Woodward, vice-president emeritus and director of the Fiftieth Anniversary Celebration. The mathematicians will meet in the last week of August and the first week of September, 1941.

One of the youngest of the great American endowed universities, the University of Chicago reaches the fiftieth anniversary of its founding recognized as an eminent pioneer in education and research.

Bibliography and Reviews

Edited by
H. A. SIMMONS

Differential and Integral Calculus. By Ross R. Middlemiss. McGraw-Hill Book Company, Inc., New York, 1940. x+416 pages; $2.50.

This book is written in the "standard" tradition as to content and order. The book is well stocked with problems of the usual type. The appendix contains more extensive tables than are customarily found in calculus textbooks.

After the author's rather careful discussion of the notion of limit and of continuity, this reviewer was disappointed to find a lamentable lack of rigor in the rest of the book. To assume, for example, as the text does, that continuity of a function insures the existence of a derivative is unfortunate, but to ask the student "why" this is so (p. 35) is, to say the least, unfair.

The treatment of the definite integral leaves too much to be desired, even in a first course, and the presence in particularly objectionable form of the much criticized "Duhamel's Principle" (without an attempt at proof) is not to be applauded.

Some readers will be disturbed by the statement of Rolle's Theorem, which is here deprived of some of its virility by the requirement that both the function and its derivative be continuous on the closed interval. Since no proof of this theorem is offered, there would seem to be no reason for failing to state this basic theorem in its most useful form.

It is only fair to observe that the errors in this text, of which those mentioned above are samples, are common to a great many books which are now in use and that this book is no doubt as good as a large number of these treatises and undoubtedly superior to some of them.

Rice Institute. WALTER LEIGHTON.

Introductory Business Mathematics. By Joel S. Georges and William H. Conley. Henry Holt and Company, Inc. New York, 1940. x+326 pages (with Appendix).

This text is certain to attract attention because of its application to business problems of such topics as logarithms and the slide rule, slope of a straight line, instantaneous rate of change, measures of central tendency, area under a curve, empirical exponential formulas, and the nomogram of $z=uxy$. Although the book is planned for the non-specializing or terminal student in the business department of the junior college, the content seems adequate for many other purposes, including the preparation of teachers of mathematics.

Annuities and life insurance are briefly treated in Part II, but there is no intention that a course from this text replace a course in mathematics of finance. Rather this course should provide a much better preparation for mathematics of finance than the customary college algebra. The authors state that the materials were selected after a survey of needs of students of business. The fact that in this survey the results of a study made by a committee of the Men's Mathematics Club of Chicago and Metropolitan Area were used is an excellent recommendation for the book. Topics are cho-

sen because of their application in business problems. The organization is according to methods, with the headings: Part I, Arithmetic Methods; Part II, Algebraic Methods; Part III, Graphic Methods. There is, without question, sufficient material for a five-hour course or two three-hour courses.

This reviewer would prefer to see the algebraic methods used earlier, even though Part I is intended as a review. Any experience with college freshmen indicates the need for a review of arithmetic, but much review could be given along with, and as part of, the later content of the course. Too, the study of the equation might be introduced through the use of the formula rather than followed by it. Such rearrangements of topics are not, of course, out of the question for the one who uses the text. The listing of many rules, sixty-five in Part I, is not in accord with a mathematical point of view in teaching modern arithmetic. If the work on formulas and equations preceded or accompanied Part I, many rules could well be eliminated, and others could be more readily established.

This text, nevertheless, is, in the reviewer's opinion, an outstanding contribution. It may have a most wholesome effect on the content of all freshmen mathematics courses. The application of algebraic and graphic methods to business problems is made in a way that provides excellent materials, not only for courses for students of business, but also for all first year college mathematics courses.

Specific good features include the well-written, concrete introductory paragraphs with each chapter; selection of drill materials; discussion exercises and the cumulative reviews at the ends of chapters; the section on the solution of equations; discussion of abbreviated multiplication and division; and the selection of content and the presentation in the chapter on the compound interest law.

Part III, Graphic Methods, deserves special mention. This section is treated in four chapters: comparison graphs, line graphs, curves, and nomography. The authors take full advantage of their opportunities to expand these topics and in so doing have compiled a good treatment with a relatively new viewpoint for business arithmetic.

Along with the bar and circle graphs, the treatment of mean, median, and mode comes in the first chapter. The analysis of line graphs permits the introduction of such concepts as continuous variable, rate of change, and slope of a line. In the third chapter of Part III, the authors define the slope of the tangent to a curve in terms of instantaneous rate of change. The use of trapezoids in finding the approximate area under a curve is introduced and it is emphasized that the area under the average price graph represents the cost. Logarithmic scales are introduced for use in curve fitting and in nomography. In an introductory paragraph the statement is made that the possibilities of nomography are being realized and appreciated by business statisticians. The twenty pages of the last chapter, chapter twelve, are devoted to this fascinating topic.

Southern Illinois State Normal University. JOHN R. MAYOR.

Elements of Analytic Geometry, by Clyde E. Love. Second Edition. The MacMillan Company, New York, 1940. xii+140 pages; $1.75.

This text is a new abridgment of the third edition of the author's *Analytic Geometry* (1938) rather than a revision of his earlier *Elements*. It is designed for a brief course, and contains about as much material as can be taught in the average class in a semester; whereas the larger work offers a more extensive treatment of the same subject matter.

The book is written with the clarity and convincing simplicity that characterize the author, and should be eminently teachable. One finds that in the earlier chapters

the material of the 1938 book is retained, with only occasional omission of a paragraph, and a few rewritten passages. Beginning with the chapter on coordinate transformations, the omissions become more drastic. The treatment of the general equation of the second degree, while adequate, is reduced to a minimum; tangents and normals are discussed in the text for the parabola only, with other curves relegated to exercises; and diameters are omitted. The excellent chapter on algebraic curves, designed as preparation for the calculus, has been recast. Parametric equations and polar coordinates are treated very briefly. There is more material on solid analytic geometry than one would expect.

The development follows traditional lines for the most part. A somewhat unusual feature is the definition of the conic sections in terms of focus and directrix instead of the more familiar separate definitions of the three types of curves. This approach is attractive from the point of view of unity but has some disadvantages. The determination of the equation is carried out separately for the parabola and for the central conics, and involves an awkward discussion to locate the center in the latter case. Further, the circle and the degenerate case of parallel lines do not fall naturally within this definition, but must be introduced as limiting cases.

The reviewer is one of those who do not believe in introducing the methods of the calculus in a first course in analytic geometry. Professor Love, like many writers on analytic geometry, derives equations of tangents to conics by the standard method of increments. Is it not better to postpone this to a course in calculus? Equations of the tangents to the conics in standard form can be neatly derived by merely getting the slope of a secant through two points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ and then bringing P_2 to P_1. The equation of the secant is itself not without interest.

For a minimum course, this text should give satisfaction, without diverting the student's attention to extraneous matters. If one desires to have additional material available for the more ambitious student, the larger text of 1938 will be more acceptable.

Brooklyn College. ROGER A. JOHNSON.

Introduction to the Calculus. By Arnold Dresden. New York: Henry Holt & Company, 1940, xii+428 pp.

Underlying the construction of this new text is a thesis and an article of faith. The thesis is that a prerequisite for the *intelligent* use of the calculus is "a thorough understanding of the fundamental principles, a knowledge of the exact conditions under which the formal processes are applicable; their limitations as well as their scope." The article of faith is that American sophomores are capable of acquiring such an understanding; that the difference in ability between American and European students of approximately equal chronological age is not accurately reflected in the difference in pedagogical standards; that a more rigorous approach will increase not only the power but even the enjoyment of the students.

This article of faith is attractive; the thesis is indisputable; and, in this reviewer's opinion, Professor Dresden has done admirably well in composing a text that can justify his faith. For admitting that the American sophomore *can* appreciate (and even enjoy!) a more rigorous development, only an exposition prepared with great care and with pedagogical insight could make this possible. And this text bears the evidence of both the care and the insight.

Derivatives are not introduced until Chapter III, functions only in Chapter II, Chapter I being devoted to real numbers (where the Dedekind cut is defined), point sets (with the Bolzano-Weierstrass theorem), and some fundamental limit theorems.

This reviewer is not entirely happy about some of the limit theorems where the notion of a function is implicit (Professor Dresden speaks in terms of monotonic variation with time) without having been formally introduced. Nevertheless, the instructor who feels the same way can skip to Chapter II before taking up these sections, by furnishing the necessary continuity in class lectures.

Functions and continuity are introduced in the second chapter with a number of examples, and additional limit theorems, including the Cauchy criterion, are presented. Derivatives are introduced, first dynamically and then geometrically, with discussion of the function $x \sin (1/x)$. Chapter IV on "Technique of Differentiation" gets to the derivative of a polynomial, but trigonometric functions are left for Chapter VII, and exponentials and logarithms for Chapter VIII.

Integration, introduced in chapter XII with the definite integral, is treated in much the same style. Multiple integrals are included with an apology (contained in the preface) as belonging more properly to advanced calculus, and the book closes with two chapters, XX and XXI, on ordinary differential equations.

Examples and applications discussed in the text are numerous and skilfully presented. Indeed, it is this that principally serves to make the crucial first three chapters convincing. In addition there is a large number of problems for the student. It will be a distinct pleasure to try out the book in class.

ALSTON S. HOUSEHOLDER.

Portraits of Famous Philosophers Who Were Also Mathematicians. With bibliographical accounts by Cassius J. Keyser, *Scripta Mathematica*, New York, 1939. 12 folders; $3.00.

With an extremely broad knowlege and appreciation of philosophers and with an enthusiasm as fresh and spontaneous as that of a sports-writer who voluntarily selects all-star baseball nines or football elevens, Professor Keyser has selected his all-star philosophy twelve, namely (with some approximations as to dates): Pythagoras (582-500 B. C.); Plato (427-347 B. C.); Aristotle (384-322 B. C.); Epicurus (341-270 B. C.); Roger Bacon (1214-1292); Rene Descartes (1596-1650); Blaise Pascal (1623-1662); Benedict Spinoza (1632-1677); Gottfried Wilhelm Leibniz (1646-1716); George Berkeley (1684-1753); Immanuel Kant (1724-1804); Charles Sanders Peirce (1839-1914).

For each philosopher, Professor Keyser gives a portrait, of quarto size, which is included in a four-page double sheet, containing a biography of roughly 1500 words. All of these portraits except two, those of Pythagoras and Roger Bacon, are genuine; these two are, of course, representations of artists' conceptions. The biographies are well-rounded accounts that enable a reader to picture the 12 heroes among their respective contemporaries.

While indicating that each hero possessed some mathematical characteristics, Professor Keyser is conservative enough in his appraisal of the heroes' mathematics. For example, he states frankly that Epicurus disliked mathematics, but that he used in his philosophy at least an embryonic concept of *infinity*. Mathematicians would surely agree with Professor Keyser that Pythagoras, Descartes, Pascal, and Leibniz were seminal thinkers in mathematics.

Since it is generally granted that the deepest inspiration in a subject comes from contact with the great masters of the subject and since the portraits and bibliographies of Professor Keyser thrust one into the closest contact with these masters, we need not argue the point that these portraits and biographies should be deeply appreciated

and widely received by all philosophers and a large group of mathematicians. The value of the biographies to any scientist will be a rapidly increasing function of the amount of study that he devotes to them.

The Keyser all-stars' portraits and biographies should be in all scientific libraries.

Northwestern University. H. A. SIMMONS.

Living Mathematics. By Ralph S. Underwood and Fred W. Sparks. McGraw-Hill Book Co., New York, 1940. x+365 pages.

The topics which Professors Underwood and Sparks admit into the category defined by the title of their book are those of elementary algebra, logarithms, progressions, theory of investment; as well as trigonometric solution of triangles, both right and oblique, analytic geometry of straight lines, of circles, and of conics in standard forms, solution of equations by successive graphing, basic notions of differential and integral calculus with applications to algebraic functions, and further assorted topics such as probability, induction, and congruences. The semicolon in the preceding sentence marks the topical end of the first part of the book and of its first one hundred and seventy-two pages, the remainder of the sentence and of the book being devoted to the material of the second part. By happy coincidence, or wise design, the first part of the book is thus precisely what the first semester college freshman of limited mathematical experience might, if left to himself, be particularly disposed to consider dead. The second part, on the other hand, is a survey of some four or five more semesters to which, after the resurrection of his surprised interest, he may be expected to turn during the rest of the year, eager for more fare of the same stimulating sort. The needs of such students have in fact been the inspiration for the book.

The principal theme of the book is palatability. Its authors admit this frankly if indirectly in their preface, and return to it thereafter with notable consistency. The titles of more than half of the chapters and of at least two-thirds of the paragraphs might be titles from a Dickens novel. Not included among these fractions are other headings ambiguously tecnnical, such as "Curves ahead", and "Table technique". In the table of contents titles such as these, it may be presumed, will appear a welcoming door into the abundant life within, displacing the barricade of unfamiliar terms generally found there. Of course the technical expressions indigenous to the subjects treated in the book occur in their proper places in its pages and in an adequate index at its end, but they do not stand at the beginning as well to discourage the timid.

Answering in their own way their own question "whether a light and jaunty treatment is suitable for a book with an essentially serious aim", the authors invigorate their material with stories somewhat better than a teacher's usual fund, appropriate historical data, witticisms, speculation, and a variety of stylistic devices. In consequence their average of symbols per page is markedly lower than it is in ordinary mathematical texts, and this would have resulted in a fairly blocky and paragraphic style if it had not been balanced by breeziness. It is to be regretted that breeziness appears to be allied to a monotonous use of personal pronouns. Moreover to teachers of the classic mode some of this breeziness may seem downright draughty, but Underwood and Sparks rarely lose sight of the fact that they are writing a book to appeal first of all to students. Their success must ultimately be gauged by the response from this section of the college population.

It may be of interest to mention not only what topics but also what aspects of conventional mathematical presentation qualify for admission into the book. Among these are *axioms* and *definitions* stated in familiar fashion, although sometimes referred

to as "bothersome but necessary formalities". Significant conclusions are rarely designated as *theorems*, although the term does occur occasionally. These three terms are presumably surface outcroppings of the underlying rigorous structure which the authors believe that they have preserved. The illustrations are all of the familiar diagrammatic form customary in mathematics. It is to be hoped that this strict abstention is intended as a reproof to those who consider that works of the kind in question may be suitably enlivened by cuts from old texts depicting the uses of surveying, or by marginal drawings none too aptly suggested by the context to some artist friend of the author. The exercises are probably ample enough to meet the likely uses of the book. Many of the lists are subdivided into "Practice Problems" and "Mettle Testers". Experiments in making homework appealing are certainly to be applauded, and those of Underwood and Sparks promise to be more effective than such current devices as the use of forbidding black stars or as calling some list of problems a "Resting Place". In the appendix are such tables as would normally be required by the subjects in the book, and a list of answers for the odd numbered problems.

The publishers are to be congratulated upon the very attractive appearance of the book.

Louisiana State University N. E. RUTT..

CORRIGENDA

In the seventh line from the bottom of page 450, (Vol. XIV), read "120" in place of "180".